Fuzzy Relations

Cartesian products, composition of relations, an equivalence properties.

Crisp Set: Cartesian Product

The Cartesian product of two universes X and Y is determined as

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

which forms an ordered pair of every $x \in X$ with every $y \in Y$, forming unconstrained matches between X and Y. Every element in universe X is related completely to every element in universe Y.

Similarly, Cartesian product (or cross product) of the sets A and B denoted

A x B is the set of ordered pairs

$$A \times B = \{(a,b) \mid a \in A, \; b \in B\}$$

Example:

Given A={1,2,3} and B={1,2} then the cartesian product of set A and B is

$$A \times B = \{(1, 1), (1, 2), (2, 1), (2,2), (3, 1), (3,2)\}$$

Fuzzy Set : Cartesian product and co-product

Let A and B be fuzzy sets in X and Y, respectively. The cartesian product of A and B, denoted by *A x B*, is a fuzzy set in the product space *X x Y* with the membership function.

$$\mu_{AxB}(x, y) = \min(\mu_A(\mathrm{x}), \mu_B(\mathrm{y}))$$

EXAMPLE: Let two fuzzy sets A = {$1/a_1$, $0.6/a_2$,$0.3/a_3$}, and B = {$0.6/b_1$, $0.9/b_2$,$0.1/b_3$}.

- A × B = { $0.6/(a_1,b_1)$, $0.9/(a_1,b_2)$, $0.1/(a_1,b_3)$, $0.6/(a_2,b_1)$, $0.6/(a_2,b_2)$, $0.1/(a_2,b_3)$, $0.3/(a_3,b_1)$, $0.3/(a_3,b_2)$, $0.1/(a_3,b_3)$}

Similarly, the Cartesian Co-product of fuzzy set A and b is denoted as A+B is fuzzy set with membership function

$$\mu_{A+B}(x, y) = \max(\mu_A(\mathrm{x}), \mu_B(\mathrm{y}))$$

Crisp Relations

- A subset of the Cartesian product $A_1 \times A_2 \times \cdots \times Ar$ is called an *r-ary* relation over $A_1, A_2, \ldots, A_r$. The most common case is for $r = 2$; the relation is a subset of the Cartesian product $A_1 \times A_2$ (i.e., a set of pairs, the first coordinate of which is from A_1 and the second from A_2). This subset of the full Cartesian product is called a binary relation from A_1 into A_2.

- If three, four, or five sets are involved in a subset of the full Cartesian product, the relations are called *ternary, quaternary,* and *quinary*.

Crisp Relation

- A relation among crisp sets A_1, A_2, …., A_n is a subset of the Cartesian product. It is denoted by R.

$$R \subseteq A_1 x\ A_2\ x \ldots .x\ A_n$$

- If a crisp relation R represents that of from sets A to B, for x ∈ A and y ∈ B, its membership function μ_R ((x, y)) is

$$\mu_R(x, y) = \begin{cases} 1, & (x, y) \in R \\ 0, & (x, y) \notin R \end{cases}$$

- This membership function maps A x B to set {0,1}

Crisp Relation

Example : Let X={ 1, 4, 5} and Y= { 3, 6,7}

Classical matrix representation for the crisp relation when

R = {(x , y)|x < y; x ∈ X, y ∈ Y} is

$$R = \begin{array}{c} \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Fuzzy Relations

- Let A, B be crisp sets, then Fuzzy relation *R* on *A* x B has degree of membership whose value lies in [0, 1].

- $$R = \{((x, y), \mu_R(x, y)) : x \in A, y \in \mathrm{B}\}$$ 'OR'
$$R = \{\mu_R(x, y)/(x, y) : x \in A, y \in \mathrm{B}\}$$
$$\mu_R : A \times B \rightarrow [0,1]$$

- The membership grade indicates strength of the present relation between elements of the tuple.
- A fuzzy relation R between two fuzzy sets A and B is a subset of the fuzzy Cartesian product A × B.

$R \subseteq A \times B$ where $A \times B$ is the fuzzy Cartesian product of the fuzzy sets A and B.

Fuzzy Relation

- Fuzzy relations are often presented in the form of two-dimensional tables (matrix representation). A *m x n* matrix represents a contented way of entering the fuzzy relation R

$$R = \begin{array}{c} \\ x_1 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & \cdots & y_n \end{array} \\ \begin{bmatrix} \mu_R(x_1, y_1) & \cdots & \mu_R(x_1, y_n) \\ \vdots & \ddots & \vdots \\ \mu_R(x_m, y_1) & \cdots & \mu_R(x_m, y_n) \end{bmatrix} \end{array}$$

Fuzzy relation

Example: Let X = {1, 2, 3} and Y = { 1, 2 }

If the membership function associated with each order pair (x, y) is given by

$$\mu_R(x, y) = e^{-(x-y)^2}$$

$$R = \left\{ \frac{e^{-(1-1)^2}}{(1,1)}, \frac{e^{-(1-2)^2}}{(1,2)}, \frac{e^{-(2-1)^2}}{(2,1)}, \frac{e^{-(2-2)^2}}{(2,2)}, \frac{e^{-(3-1)^2}}{(3,1)}, \frac{e^{-(3-2)^2}}{(3,2)} \right\}$$

$$R = \left\{ \frac{1.0}{(1,1)}, \frac{0.37}{(1,2)}, \frac{0.37}{(2,1)}, \frac{1.0}{(2,2)}, \frac{0.02}{(3,1)}, \frac{0.37}{(3,2)} \right\}$$

In the second method using the relational matrix, we have

$$R = \begin{bmatrix} 1 & 0.37 \\ 0.37 & 1 \\ 0.02 & 0.37 \end{bmatrix}$$

Fuzzy Binary Relation

- Let X and Y two Universe of discourse. Then

$R = \{((x,y), \mu_R(x,y)) : (x,y) \in X \times Y\}$ in a Binary fuzzy relation in $X \times Y$

Example of binary relations

- y is greater than x (x and y are numbers)
- x is close to y (x and y are numbers)
- x depends on y (x and y are events)
- If x is large, then y is small

A fuzzy if-then rule be defined as a binary fuzzy relation R on the product space $X \times Y$

A fuzzy implication A $\longrightarrow$ B describes a relation between two variables x and y;

Extension Principle:

$$x \rightarrow \boxed{f(x)} \rightarrow y$$

Extention is mapping from x-to-y through f(·).

$y = f(x)$

$f: X \longrightarrow Y$ (From one universe to another universe)

Extension Principle (Fuzzy set):

Suppose f is a mapping *from an n-dimensional Cartesian product* space $X_1 \times X_2 \times \ldots \times X_n$ to a *one-dimensional universe Y* such that and suppose $A_1, A_2, \ldots, A_n$ are n fuzzy sets in $x_1, x_2, \ldots, x_n$ respectively. Then, the image of $A_1, A_2, \ldots, A_n$ under ***f*** is given as:

$$\mu_B(y) = \max_{f(x_1, x_2, \ldots xn) = y} (\,\mu_A(x)\,)$$

Extension Principle (Fuzzy set):

Suppose that *f is a function from X to Y* and *A is a fuzzy set on X defined as*

$$A = \frac{\mu_A(x1)}{x1} + \frac{\mu_A(x2)}{x2} + \dots + \frac{\mu_A(xn)}{xn}$$

The extension principle states that the image of fuzzy set *A under the mapping f(.) can be expressed as a fuzzy set B.*

$$B = f(A) = \frac{\mu_A(x1)}{y_1} + \frac{\mu_A(x2)}{y_2} + \dots + \frac{\mu_A(xn)}{y_n}$$

where $y_i = f(x_i)$

Extension Principle (Fuzzy set):

Example : Let $A = 0.1/-2 + 0.4/-1 + 0.8/0 + 0.9/1 + 0.3/2$
and
$f(x) = x^2 - 3$

Then $B = 0.1/1 + 0.4/-2 + 0.8/-3 + 0.9/-2 + 0.3/1$
$= 0.8/-3 + (0.4 \vee 0.9)/-2 + (0.1 \vee 0.3)/1$
$= 0.8/-3 + 0.9/-2 + 0.3/1$

Cylindrical Extension

- Assume X and Y are two crisp sets and let A be a fuzzy subset of X. The cylindrical extension of A to X × Y, denoted by $\hat{A}$ = A× Y, is a fuzzy relationship on X × Y.

$$\hat{A}(x,y) = A(x) \wedge Y(y) = A(x) \wedge 1 = A(x), \quad (x,y) \in X \times Y$$

Assume X={a, b, c} and Y={1, 2}.

Let A={1/a, 0.6/b, 0.3/c}.

Then the cylindrical extension of A to X × Y is

{1/(a,1), 1/(a,2), 0.6/(b,1), 0.6/(b,2), 0.3/(c,1), 0.3/(c,2)}

Projection

Assume A is a fuzzy relationship on X × Y.

The projection of A onto X is a fuzzy subset A^o of X, denoted by

$$A^o = \mathrm{Proj}_x A, \quad \mathrm{Ao}(\mathrm{x}) = \max_{y}[A(x, y)]$$

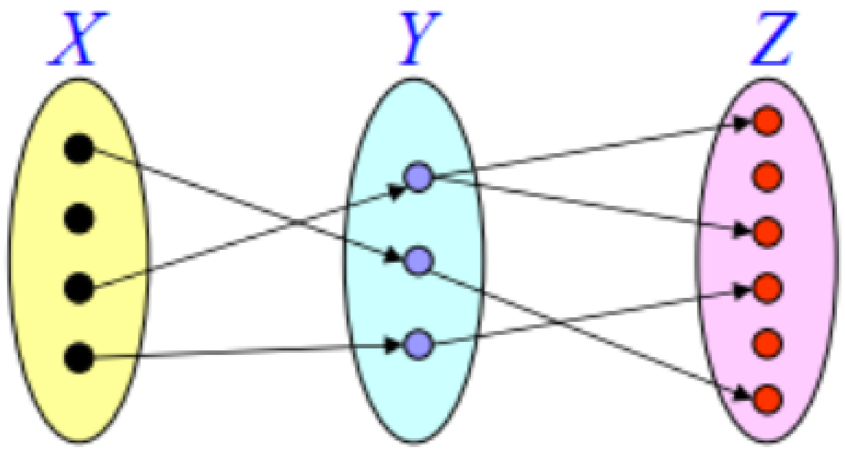
Assume X = {a,b,c} and Y = {1,2}.

Let A={1/(a,1), 0.6/(a,2), 0.8/(b,1), 0.6/(b,2), 0.3/(c,1), 0.5/(c,2)}.

Then Proj_x A = {1/a, 0.6/b, 0.5/c}.

Proj_yA = {0.8/1, 0.6/2}.

Max-Min Composition Relation



R: fuzzy relation defined on X and Y.

S: fuzzy relation defined on Y and Z.

$R \circ S$: the composition of R and S.

A fuzzy relation defined on X an Z.

$$\mu_{R\circ S}(x,z) = \max_y \min\left(\mu_R(x,y), \mu_S(y,z)\right)$$

$$= \vee_y \left(\mu_R(x,y) \wedge \mu_S(y,z)\right)$$

Max-Min Composition Relation

$$\mu_{S\circ R}(x, y) = \max_{v} \min\left(\mu_R(x,v), \mu_S(v,y)\right)$$

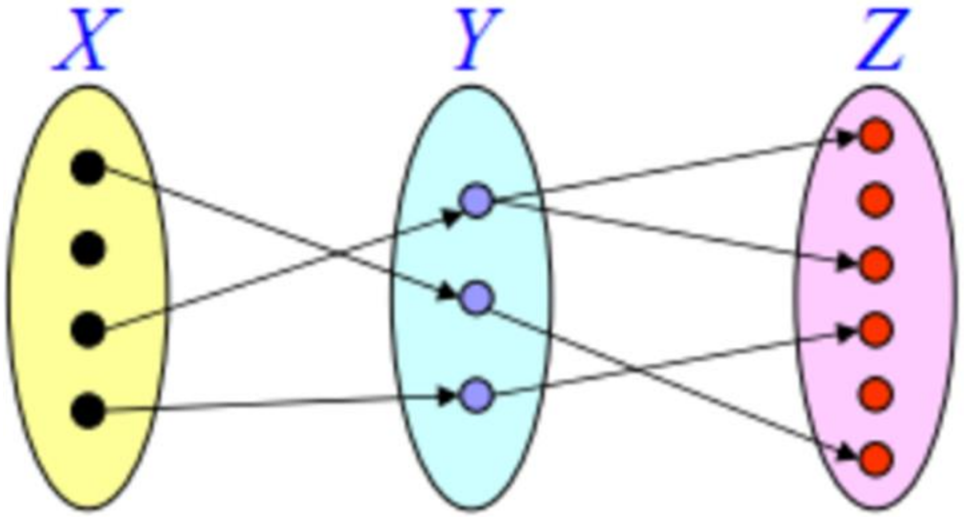
R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

	0.1	0.2	0.0	1.0
min	0.9	0.2	0.8	0.4
max	(0.1	0.2	0.0	0.4)

$R \circ S$	α	β	γ
1	0.4	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

Max-Product Composition Relation



R: fuzzy relation defined on X and Y.

S: fuzzy relation defined on Y and Z.

$R \circ S$: the composition of R and S.

A fuzzy relation defined on X an Z.

$$\mu_{R \circ S}(x, y) = \max_{v}\left(\mu_R(x, v) \bullet \mu_S(v, y)\right)$$

Operations on Fuzzy Relations

Let $\underset{\sim}{R}$ and $\underset{\sim}{S}$ be fuzzy relations on the Cartesian space $X \times Y$. Then the following operations apply for the membership values for various set operations:

Union $$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y))$$

Intersection $$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y))$$

Complement $$\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y)$$

Containment $$\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y)$$

Crisp tolerance and equivalence relation

A relation R on a universe X can also be thought of as a relation from X to X. The relation R is an equivalence relation if it has the following three properties: (1) reflexivity, (2) symmetry, and (3) transitivity.

For example, for a matrix relation the following properties will hold:

Reflexivity $(x_i, x_i) \in R$ or $\chi_R(x_i, x_i) = 1$

Symmetry $(x_i, x_j) \in R \longrightarrow (x_j, x_i) \in R$

or $\chi_R(x_i, x_j) = \chi_R(x_j, x_i)$

Transitivity $(x_i, x_j) \in R$ and $(x_j, x_k) \in R \longrightarrow (x_i, x_k) \in R$

or $\chi_R(x_i, x_j)$ and $\chi_R(x_j, x_k) = 1 \longrightarrow \chi_R(x_i, x_k) = 1$

Crisp tolerance relation

A tolerance relation *R* (also called a proximity relation) on a universe *X* is a relation that exhibits only the properties of reflexivity and symmetry. A tolerance relation, R, can be reformed into an equivalence relation by at most *(n − 1)* compositions with itself, where *n* is the cardinal number of the set defining *R*, in this case X, i.e.,

$$R_1^{n-1} = R_1 \circ R_1 \circ \cdots \circ R_1 = R$$

Fuzzy Tolerance and Equivalence Relations

A fuzzy relation, R , on a single universe X is also a relation from X to X. It is a fuzzy equivalence relation if all three of the following properties for matrix relations define it:

Reflexivity $\quad \mu_{\underset{\sim}{R}}(x_i, x_i) = 1$

Symmetry $\quad \mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i)$

Transitivity $\quad \mu_{\underset{\sim}{R}}(x_i, x_j) = \lambda_1 \quad \text{and} \quad \mu_{\underset{\sim}{R}}(x_j, x_k) = \lambda_2 \longrightarrow \mu_{\underset{\sim}{R}}(x_i, x_k) = \lambda$

where $\lambda \geq \min[\lambda_1, \lambda_2]$.

Fuzzy Tolerance Relations

Fuzzy tolerance relation, R_1, that has properties of reflexivity and symmetry can be reformed into a fuzzy equivalence relation by at most *(n − 1)* compositions, just as a crisp tolerance relation can be reformed into a crisp equivalence relation. That is,

$$R_1^{n-1} = R_1 \circ R_1 \circ \cdots \circ R_1 = R$$

Example: Composition Relation

Let $X = \{x_1, x_2\}$ *and* $Y = \{y_1, y_2\}$ *and* $Z = \{z_1, z_2\}$

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then find the max-min composition and max product composition

$$S = R \circ Q$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ is the max-min composition.

and $$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the max product composition

Example: Composition Relation(max-min)

Let $R_1(x,y)$ *and* $R_2(x,y)$ be defined by the following relational matrix

$$R_1 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0 & 1 & 0.7 \\ 0.3 & 0.5 & 0 & 0.2 & 1 \\ 0.8 & 0 & 1 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

$$R_2 = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{bmatrix} 0.9 & 0 & 0.3 & 0.4 \\ 0.2 & 1 & 0.8 & 0 \\ 0.8 & 0 & 0.7 & 1 \\ 0.4 & 0.2 & 0.3 & 0 \\ 0 & 1 & 0 & 0.8 \end{bmatrix} \end{array}$$

we shall first compute the max-min composition $R_1 \circ R_2(x,z)$

$$\mu_{R_1 \circ R_2}(x_1, z_1) = \max(\min(0.1, 0.9), \min(0.2, 0.2), \min(0, 0.8), \min(1, 0.4), \min(0.7, 0))$$

$$= \max(0.1, 0.2, 0, 0.4, 0) = 0.4$$

Similarly we can determine the grades of membership for all pairs

$$R_1 o R_2 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{bmatrix} 0.4 & 0.7 & 0.3 & 0.7 \\ 0.3 & 1 & 0.5 & 0.8 \\ 0.8 & 0.3 & 0.7 & 1 \end{bmatrix} \end{array}$$

Composition Relation (max-product)

for the max product composition, we calculate

$$\mu_{R_1}(x_1, y_1).\mu_{R_2}(y_1, z_1) = 0.1.0.9 = 0.09$$

$$\mu_{R_1}(x_1, y_2).\mu_{R_2}(y_2, z_1) = 0.2.0.2 = 0.04$$

$$\mu_{R_1}(x_1, y_3).\mu_{R_2}(y_3, z_1) = 0.0.8 = 0$$

$$\mu_{R_1}(x_1, y_4).\mu_{R_2}(y_4, z_1) = 1.0.4 = 0.4$$

$$\mu_{R_1}(x_1, y_5).\mu_{R_2}(y_5, z_1) = 0.7.0 = 0$$

hence

$$\mu_{R_1 o R_2}(x_1, z_1) = \max\{0.09, 0.04, 0, 0.4, 0\} = 0.4$$

In the similar way after performing the remaining computation, we obtain

$$R_1 o R_2 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{bmatrix} 0.4 & 0.7 & 0.3 & 0.56 \\ 0.27 & 1 & 0.4 & 0.8 \\ 0.8 & 0.3 & 0.7 & 1 \end{bmatrix} \end{array}$$

Example: Composition Relation

Let $R_1(x, y)$ and $R_2(x, y)$ be defined as the following relational matrices

$$R_1 = \begin{bmatrix} 0.6 & 0.5 \\ 1 & 0.1 \\ 0 & 0.7 \end{bmatrix} \quad \text{and } R_2 = \begin{bmatrix} 0.7 & 0.3 & 0.4 \\ 0.9 & 0.1 & 0.6 \end{bmatrix}$$

We shall first calculate the max-min composition $R_1 \circ R_2$

$$R_1 \circ R_2 = \begin{bmatrix} 0.6 & 0.5 \\ 1 & 0.1 \\ 0 & 0.7 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.3 & 0.4 \\ 0.9 & 0.1 & 0.6 \end{bmatrix}$$

Now we calculate

$$\mu_{R_1 \circ R_2}(x_1, z_1) = \max\left(\min(0.6, 0.7), \min(0.5, 0.9)\right) = \max(0.6, 0.5) = 0.6$$

Similarly we can calculate the other entries. The relational matrix for max-min composition in fuzzy relation is thus

$$R_1 o R_2 = \begin{bmatrix} 0.6 & 0.3 & 0.5 \\ 0.7 & 0.3 & 0.4 \\ 0.7 & 0.1 & 0.6 \end{bmatrix}$$

Composition Relation

Given is a fuzzy relation R: $X \times Y \to [0,1]$:

$$R = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.7 & 0.3 & 0.1 \\ x_2 & 0.4 & 0.8 & 0.2 \\ x_3 & 0.1 & 0.2 & 0.9 \end{array}$$

and a fuzzy set $A = \{0.1/x_1,\ 1/x_2,\ 0.4/x_3\}$. Compute fuzzy set $B = A \circ R$, where '$\circ$' is the max-min composition operator.

Question

$$\underset{\sim}{F} = \left\{ \frac{0.4}{h_1} + \frac{0.9}{h_2} + \frac{0.6}{h_3} \right\}$$

$$\underset{\sim}{D} = \left\{ \frac{0.1}{t_1} + \frac{1.0}{t_2} \right\}$$

Find the Cartesian product $\underset{\sim}{F} \times \underset{\sim}{D} = \underset{\sim}{G}$

$$\underset{\sim}{E} = \left\{ \frac{0.2}{h_1} + \frac{1.0}{h_2} + \frac{0.3}{h_3} \right\}$$

Using a max–min composition find $\underset{\sim}{C} = \underset{\sim}{E} \circ \underset{\sim}{G}$

Questions

$$\text{Average current (in amps)} = \underset{\sim}{I} = \left\{ \frac{0.4}{0.8} + \frac{0.7}{0.9} + \frac{1}{1} + \frac{0.8}{1.1} + \frac{0.6}{1.2} \right\}$$

$$\text{Average voltage (in volts)} = \underset{\sim}{V} = \left\{ \frac{0.2}{30} + \frac{0.8}{45} + \frac{1}{60} + \frac{0.9}{75} + \frac{0.7}{90} \right\}$$

Find the fuzzy Cartesian product $\underset{\sim}{P} = \underset{\sim}{V} \times \underset{\sim}{I}$.

Now let us define a fuzzy set for the cost C

$$\underset{\sim}{C} = \left\{ \frac{0.4}{0.5} + \frac{1}{0.6} + \frac{0.5}{0.7} \right\}$$

Using a fuzzy Cartesian product, find $\underset{\sim}{T} = \underset{\sim}{I} \times \underset{\sim}{C}$

Using max–min composition, find $\underset{\sim}{E} = \underset{\sim}{P} \circ \underset{\sim}{T}$.

Using max–product composition, find $\underset{\sim}{E} = \underset{\sim}{P} \circ \underset{\sim}{T}$.