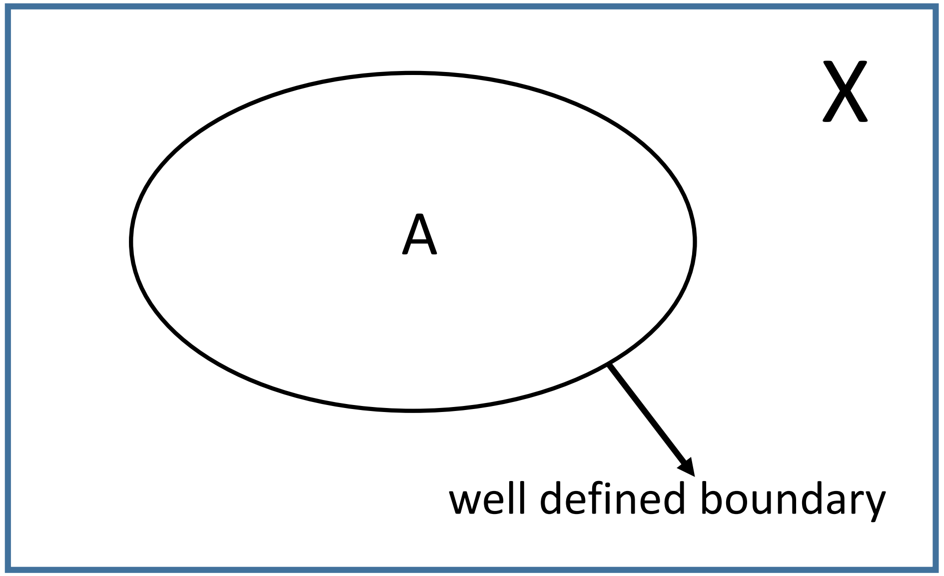
UNIT-I

Fuzzy logic, Fuzzy Set Theory, Crisp Sets, Fuzzy sets

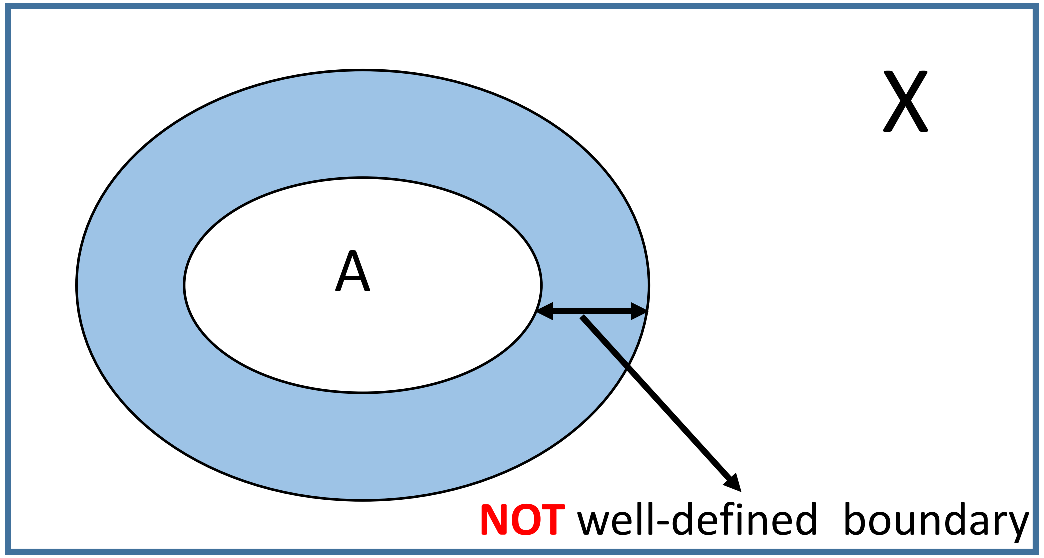
Contents

- Operation on Classical Sets
- Properties of Classical (Crisp) Sets
- Mapping of Classical Sets to Functions Fuzzy Sets
- Notation and Convention for Fuzzy Sets
- Fuzzy Set Operations

Classical Set vs Fuzzy Set



Classical (*crisp*) set theory



Fuzzy set theory

- A classical set is defined by crisp boundaries (or *clearly defined boundaries, or sharp boundaries, or well-defined boundaries*.). In classical set theory, a set possesses a well-defined boundary, This means that an element in a classical set is either a member(1) or a non-member(0).
- A fuzzy set can exhibit an arbitrary membership degree between zero and one due to its vague or ambiguous set boundaries or NOT well-defined boundaries.

Classical (Crisp) Set

- A set is a collection of objects with a common property.
- A universe of discourse, X, as a collection of objects all having the same characteristics.
- X could be the *N, Z, Q, R* depending on the context.
- For the set of the vowels of the alphabet, X would be all the letters of the alphabet.
- The individual elements in the universe X will be denoted as x_i.
- The features of the elements in X can be discrete, countable integers or continuous valued quantities on the real line. The total number of elements in a set A is called its cardinal number, denoted by $|A|$
- Collections of elements within a universe are called sets.
- Collections of elements within sets are called subsets
- The collection of all possible sets in the universe is called the whole set (power set). The power set of set A is $|P(A)| = 2^{|A|}$

Representation of Classical/Crisp Sets:

- Enumeration of elements: $A = \{x_1, x_2, \ldots, x_n\}$

- Definition by property: $A = \{x \in X \mid x \text{ has property } P\}$

- Characteristic function/Membership function: $\mu_A(x) : X \rightarrow \{0, 1\}$

$$\mu_A(x) = \begin{cases} 1, & x \text{ is member of } A \\ 0, & x \text{ is not member of } A \end{cases}$$

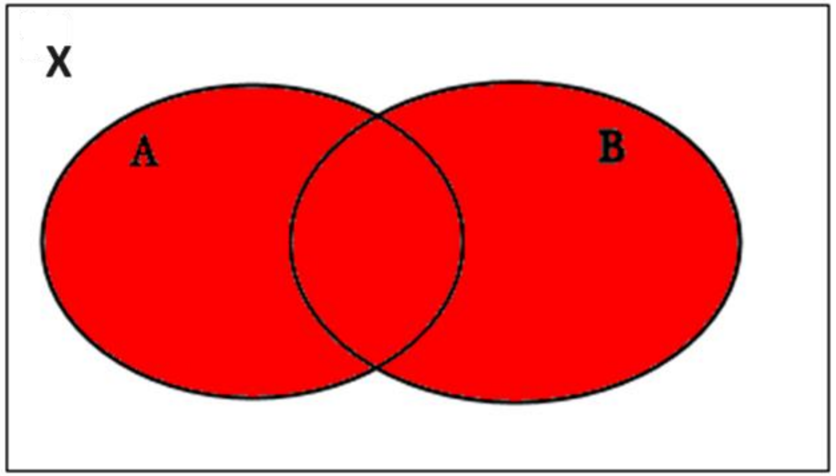
Classical (Crisp) Set: Operations

Union:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}, \text{ universe of discourse is } X$$

- Represents all those elements in the universe that reside in (or belong to) the set A, the set B, or both sets A and B.

- This operation is also called the logical or



A ∪ B

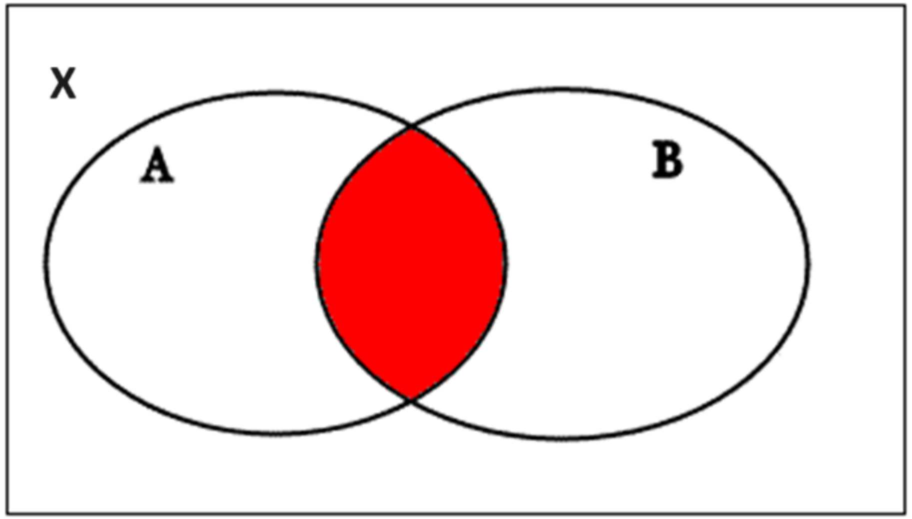
Classical (Crisp) Set: Operations

Intersection:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}, \text{ universe of discourse is } X$$

Represents all those elements in the universe X that simultaneously reside in (or belong to) both sets A and B.

- This operation is also called the logical and



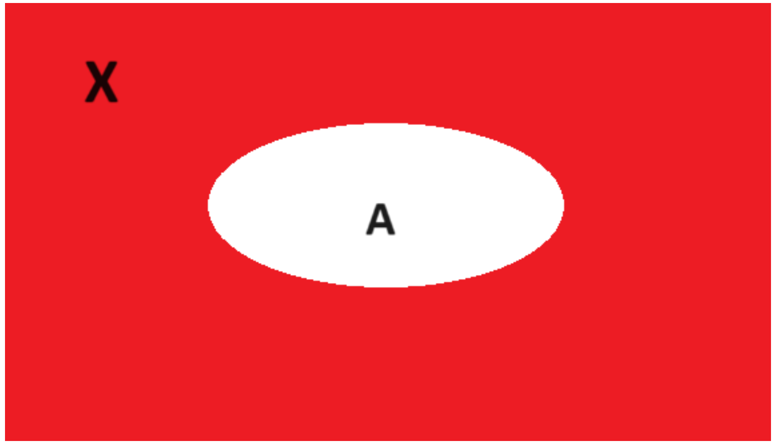
$A \cap B$

Classical (Crisp) Set: Operations

Complement:

$A' = \{x \mid x \in X \text{ but } x \notin A\}$, universe of discourse is X

Represents collection of all elements in the universe that do not reside in the set A.



Classical (Crisp) Set: Operations

Difference:

$$A - B = \{x \mid x \in A \text{ but } x \notin B\}$$

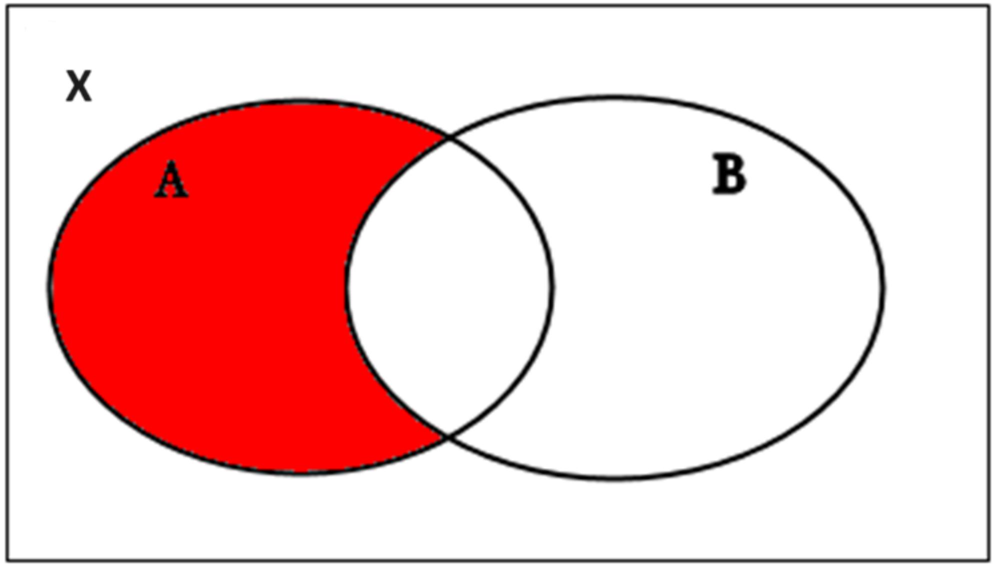
Represents the collection of all elements in the universe that reside in A and that do not reside in B simultaneously.

Similarly,

$$B - A = \{x \mid x \in B \text{ but } x \notin A\}$$

$A\text{-}B = A \cap \overline{B}$

$B\text{-}A = B \cap \overline{A}$

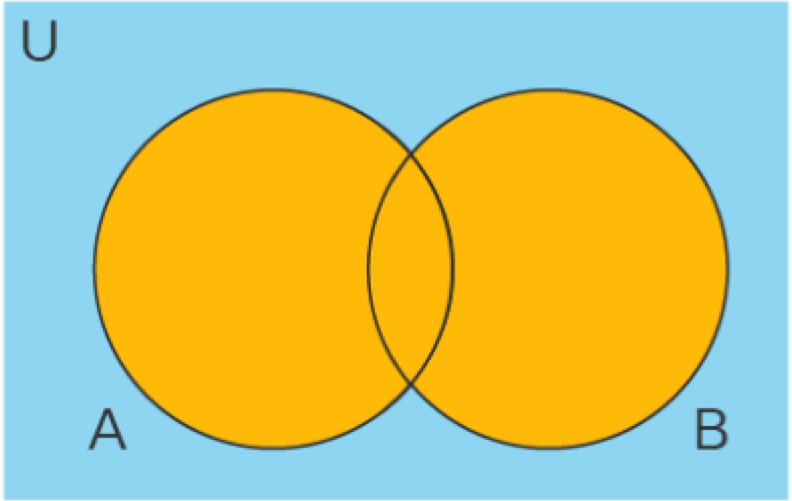


A - B

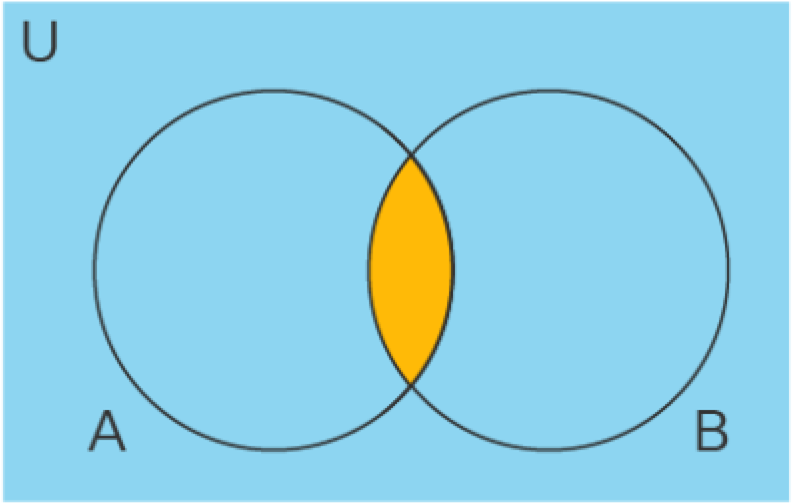
Classical (Crisp) Set: De Morgan's law

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$



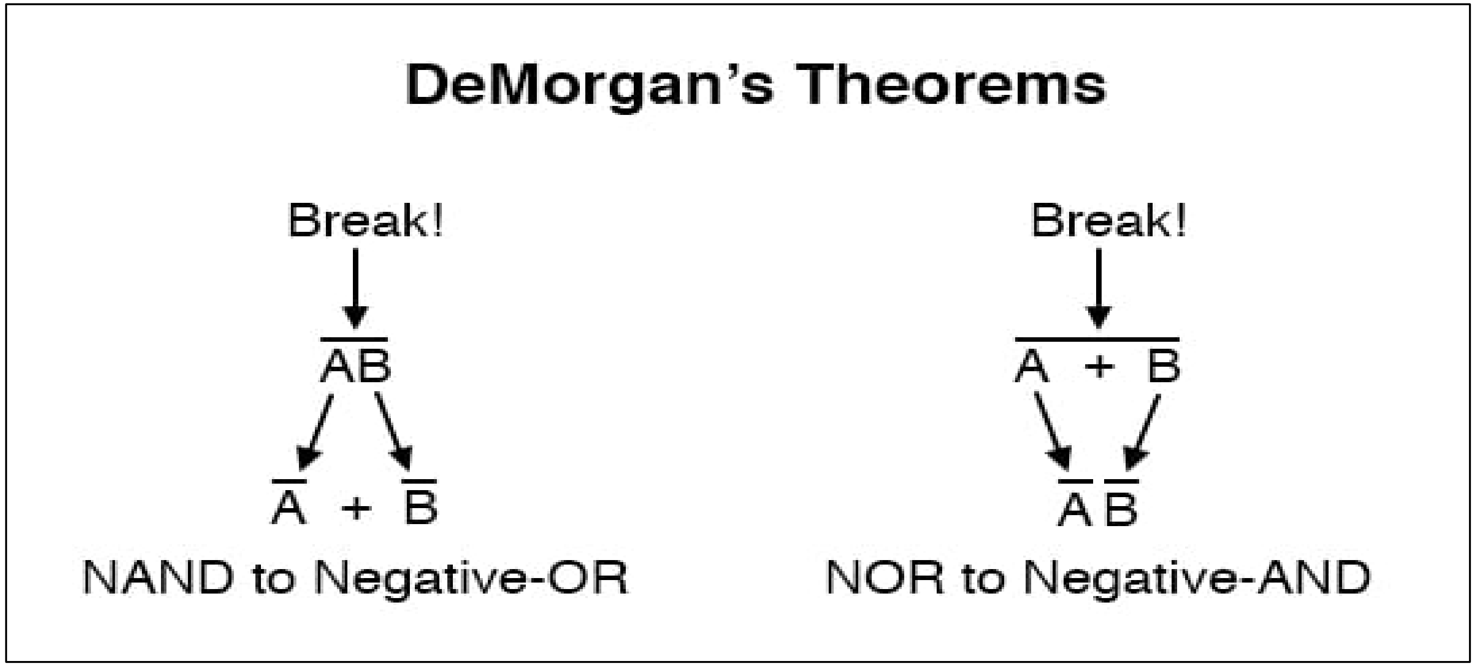
$(A \cup B)' = A' \cap B'$



$(A \cap B)' = A' \cup B'$

Classical (Crisp) Set: De Morgan's law

- De Morgan's theorem may be thought of in terms of *breaking* a long bar symbol.
- When a long bar is broken, the operation directly underneath the break changes *from addition to multiplication, or vice versa*, and the broken bar pieces remain over the individual variables.



Classical (Crisp) Set: Properties

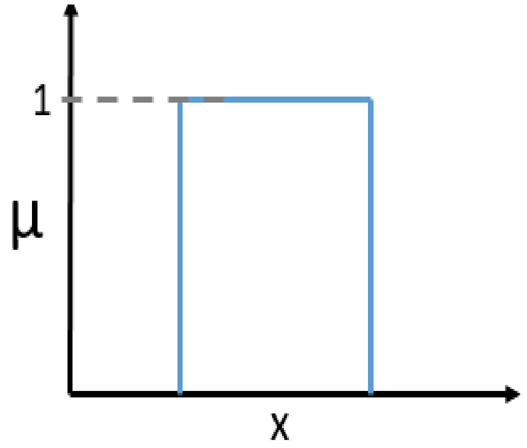
Law of contradiction	$A \cap \overline{A} = \emptyset$
Law of the excluded middle	$A \cup \overline{A} = X$
Idempotency	$A \cap A = A,\ A \cup A = A$
Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cap B = B \cap A,\ A \cup B = B \cup A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Absorption	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Absorption of complement	$A \cup (\overline{A} \cap B) = A \cup B$ $A \cap (\overline{A} \cup B) = A \cap B$
DeMorgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

NOT hold in Fuzzy set (applies to Law of contradiction and Law of the excluded middle)

Mapping of Classical Sets to Functions

- The characteristic function or membership function **μ_A** is defined by:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$



- *The characteristic function* μ_A of a crisp set A is analogous to the membership function of a fuzzy set.
- *There are only two values which represents whether an element belongs to a set (i.e. member of the set, whose membership value is 1 (one)) or not belongs to the set(i.e. not a member of the set, whose value is zero (0))*

$$\mu_A(x) : X \rightarrow \{0, 1\}$$

Why fuzzy?

- The word "fuzzy" means "vaguness (ambiguity)".

  Fuzzy →(vague, uncertain, inexact etc.)

In real world, human thinking and reasoning (analysis, logic, interpretation) frequently involved fuzzy information.

" *Fuzzy system have been provide solution*"

- Statistical models deal with *random events and outcomes*; fuzzy models attempt to capture *and quantify nonrandom imprecision.*
- *Randomness refers to an event that may or may not occur. Fuzziness refers to the boundary of a set that is not precise.*
- *Example : Randomness: frequency of car accidents, Fuzziness: the seriousness of a car accident.*

Fuzzy logic and Fuzzy set:

- *A logic based on the two truth values TRUE and FALSE is sometimes inadequate when describing human reasoning.*
- *Fuzzy logic uses the whole interval between 0 (FALSE) and 1 (TRUE) to describe human reasoning.*
- *Fuzzy logic deals with Fuzzy set.*

Definition: Let X be the universal set, The fuzzy set A in X is a set of ordered pairs; $A := \{(x; \mu_A(x)) : x \in X\}$, $\mu_A(x): X \to [0, 1]$

where, $\mu_A(x): X \to [0, 1]$ *is called the membership function. The value of* $\mu_A(x)$ *is called the grade of membership of x in A*

Definition: Let $x \in X$*, then x is called*

(i) *Not include /Not a member in the fuzzy set if* $\mu_A(x) = 0$

(ii) *Partial include/Partial Membership if* $0 < \mu_A(x) < 1$

(iii) *Full include/Full membership if* $\mu_A(x) = 1$

Formal definitions of a fuzzy set:

- For any fuzzy set, A, the function μ_A represents the membership function for which $\mu_A(x)$ indicates the degree of membership that x, of the universal set X, belongs to set A and is, usually, expressed as a number between 0 and 1.

$$\mu_A(x) : X \rightarrow [0,1]$$

- Fuzzy sets can be either discrete or continuous

Representation Methods Of Fuzzy Set

If elements are discrete, then the fuzzy set A on X can be represented by

(i) $A = \{(x, \mu_A(x)), x \in X\}$

(ii) $A = \sum_i^n \mu_A(xi)/x_i$

(iii) $A = \alpha_1/x_1 + \alpha_2/x_2 + \alpha_3/x_3 + \ldots + \alpha_n/x_n$, where $x_i \in X$, $\alpha_i = \mu_A(xi)$

(v) Graphical method

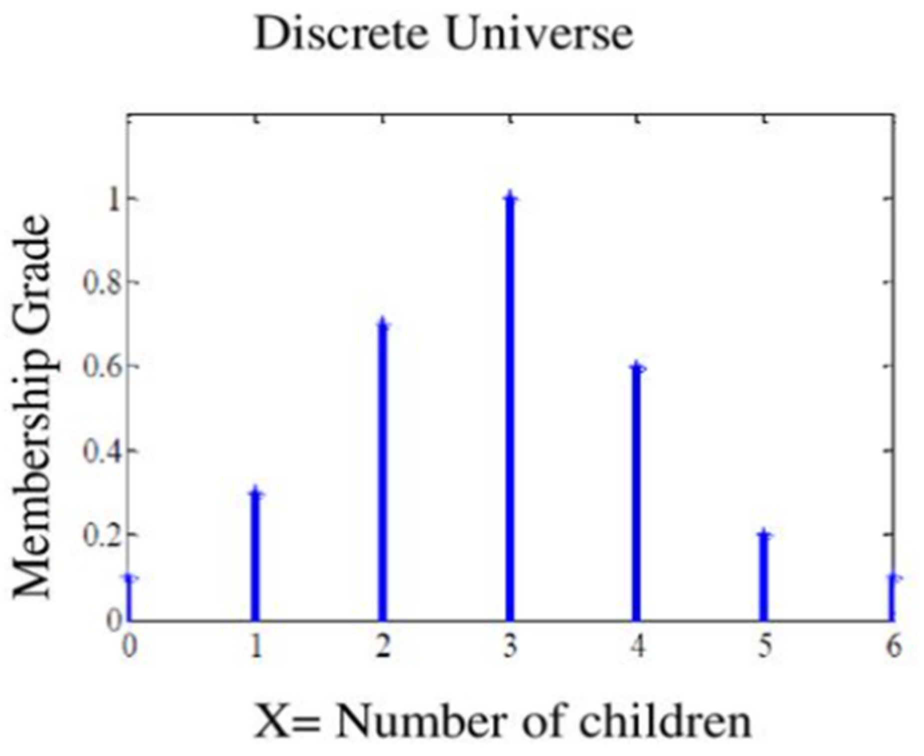
- *Where the symbol "/" is not a division sign but indicates that the top number $\mu_A(x)$ is the membership value of the element x in the bottom.*
- *Summation symbol used only for notations. This NOT actual summation*

Discrete Fuzzy Set:

$$A = \{\mu_A(x_1)/x_1, \ldots, \mu_A(x_n)/x_n\} = \{\mu_A(x_i)/x_i \mid x_i \in X\}$$

$$A = \sum_{x_i \in A} \mu_A(x_i)/x_i$$

Let X = {0, 1, 2, 3, 4, 5, 6} be a set of numbers of children a family may possibly have.
Fuzzy set A with “sensible number of children in a family” may be described by
A = {(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.6), (5, 0.2), (6, 0.1)}



Representation Methods Of Fuzzy Set

If elements are continuous, then the fuzzy set A on X can be represented by

1) $\int_X \mu_A(x)/x$

2)

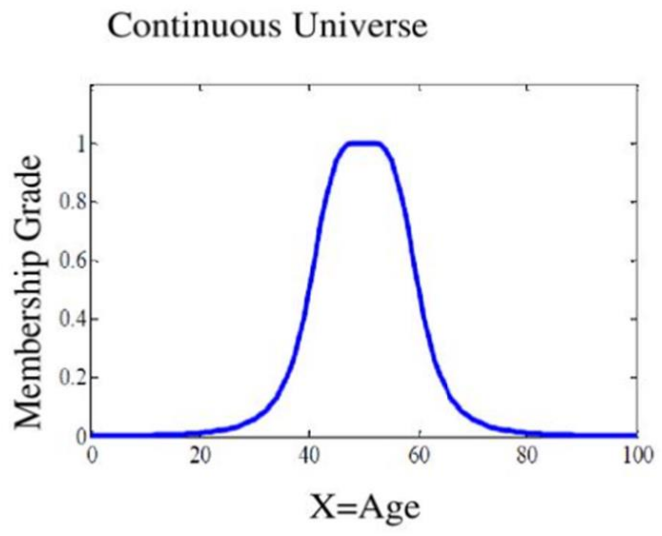
x_1	x_2	x_3	x_4	x_5
0.2	0.5	0.4	0.7	0.6

3) Graphical method

- *Where the symbol "/" is not a division sign but indicates that the top number $\mu_A(x)$ is the membership value of the element x in the bottom.*
- *Integration symbol used only for notations. This NOT actual integration*

Continuous Fuzzy Set:

$$A = \int_X \mu_A(x)/x$$

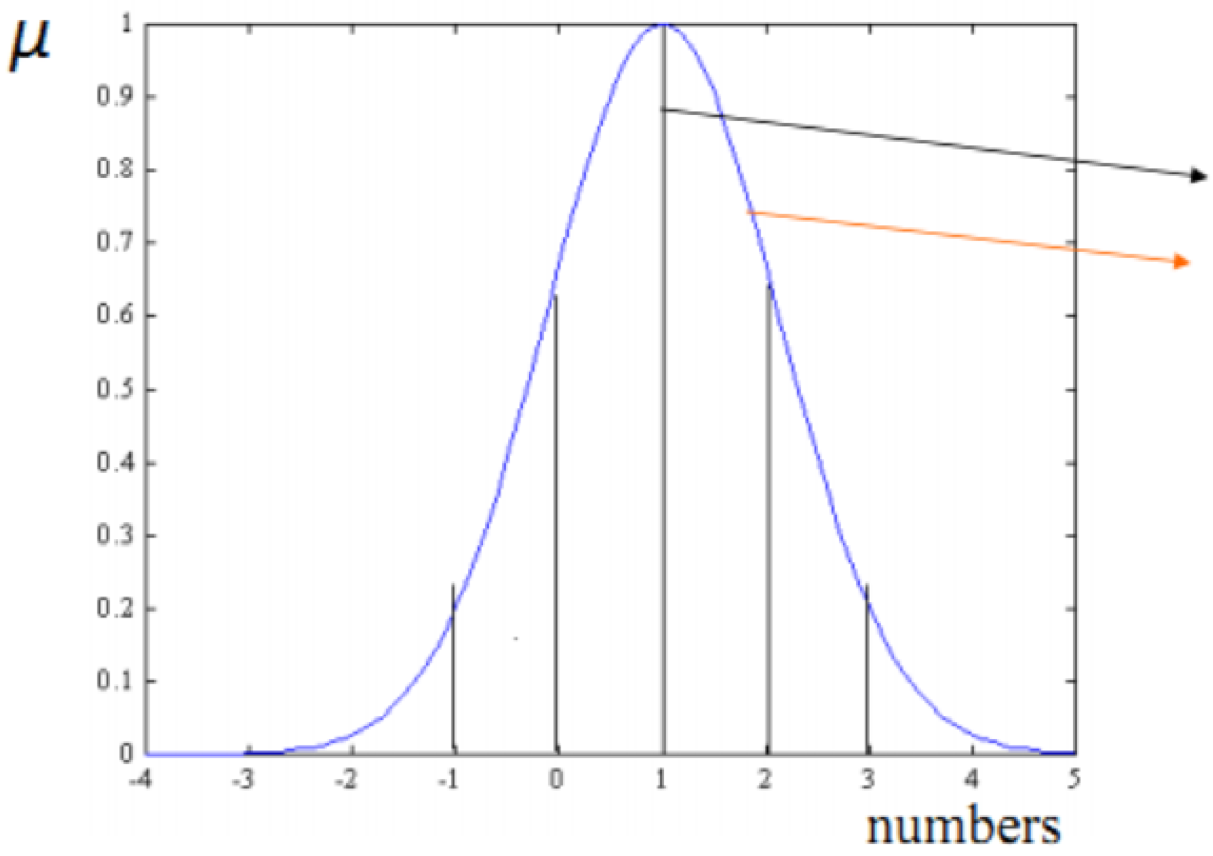


X = R+ be the set of possible ages for human beings. fuzzy set B = "about 50 years old" may be expressed as B = {(x, $\mu_A(x) | x \in X$}, where $\mu_A(x) = 1/(1 + ((x-50)/10)^4$

Question: draw the graph for the continuous functions (i) $\mu_A(x) = \frac{1}{1+x^2}$,

(ii) $\mu_A(x) = (\frac{1}{1+x^2})^2$

Fuzzy Sets(Discrete and continuous)



Example:
Discrete and
Continuous
fuzzy sets to
represent the set
of numbers
"close to 1"

Crisp set vs Fuzzy set

- Crisp set :
  - Membership function
  - Membership degree: {0,1}

A crisp set/classical set, A, the characteristic function for each element x of X can be represented by ordered pairs (x, 0) or (x, 1) , which indicates $x \notin A$ *and* $x \in A$ respectively.

- Fuzzy set :
  - Membership function: user specify
  - Membership degree: [0,1]

A fuzzy set *A* in the universe *X* is a set of *ordered pairs* $A = \{(x, \mu_A(x)), x \in X\}$ *where* $\mu_A(x)$ *is the grade of membership of x in A*

Fuzzy set:

- A classical set is defined by crisp(exact) boundaries, i.e., there is no uncertainty about the set boundaries.
- A fuzzy set is defined by its ambiguous boundaries, i.e., there exists uncertainty about the set boundaries.
- A set without crisp boundary is known as fuzzy set. The transition from "belong to a set" to "not-belong to the set" is gradual and this smooth *transition is characterized by the membership function.*
- *Fuzzy sets allows a grading of to what extent an element of a set belongs to that specific set.*
- The value 0 means that x is not a member of the fuzzy set; the value 1 means that x is fully a member of the fuzzy set.
- The values between 0 and 1 characterize fuzzy members, which belong to the fuzzy set only partially.

Fuzzy Set

- Weather is **Hot**?

  Crisp: Yes (1) ! No(0)!

  Fuzzy: Extremely Hot (1) Very Hot(0.80) warm(0.40)

  little warm(0.0)

- Words like young, tall, good are fuzzy.

There is no single quantitative value which defines the term **young**.

For some people, age 25 is young, and for others, age 35 is young.

The concept young has no clear boundary.

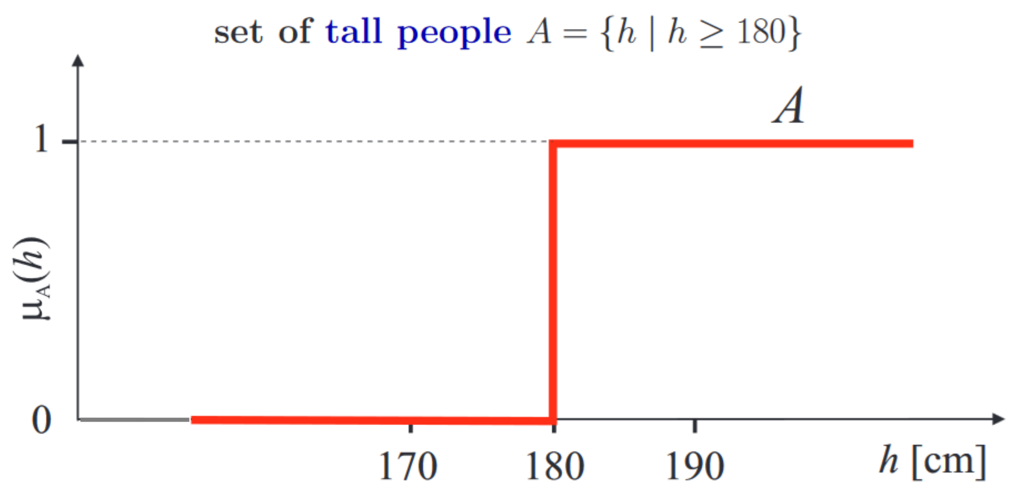
Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

Fuzzy set theory is an extension of classical set theory where elements have degree of membership/grade of membership.

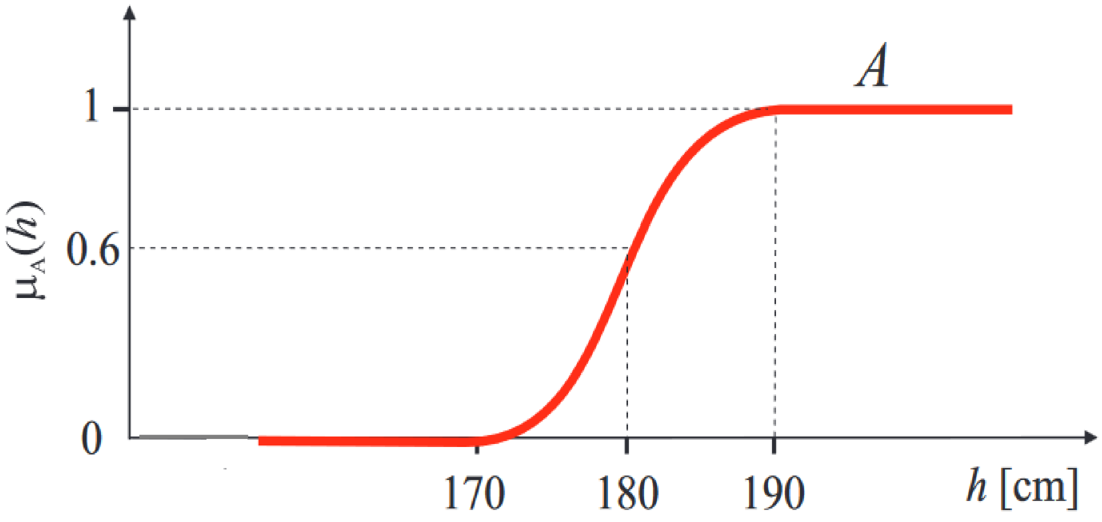
Fuzzy Set

- Set of all **tall** peoples representation in classical and fuzzy set

:Classical Set Approach:



:Fuzzy Set Approach:



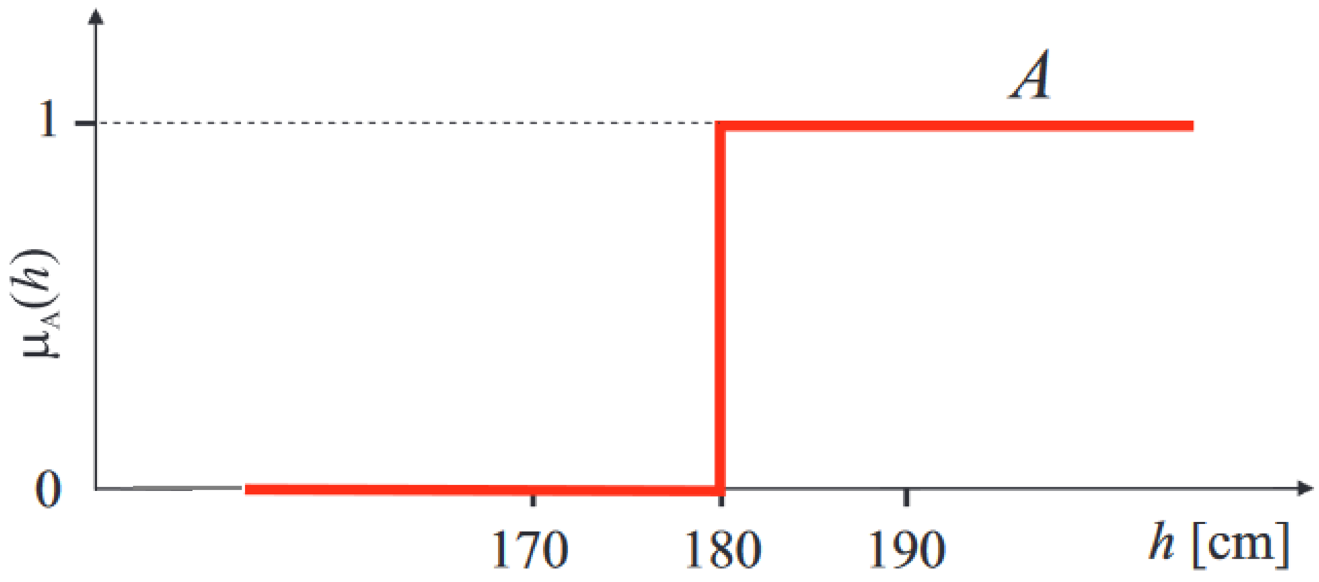
$$\mu_A(h) = \begin{cases} 1 & h \text{ is full member of } A \quad (h \geq 190) \\ (0,1) & h \text{ is partial member of } A \quad (170 < h < 190) \\ 0 & h \text{ is not member of } A \quad (h \leq 170) \end{cases}$$

Logical Propositions

"John is tall" . . . true or false

John's height: $h_{John} = 180.0$ $\mu_A(180.0) = 1$ **(true)**

$h_{John} = 179.5$ $\mu_A(179.5) = 0$ **(false)**



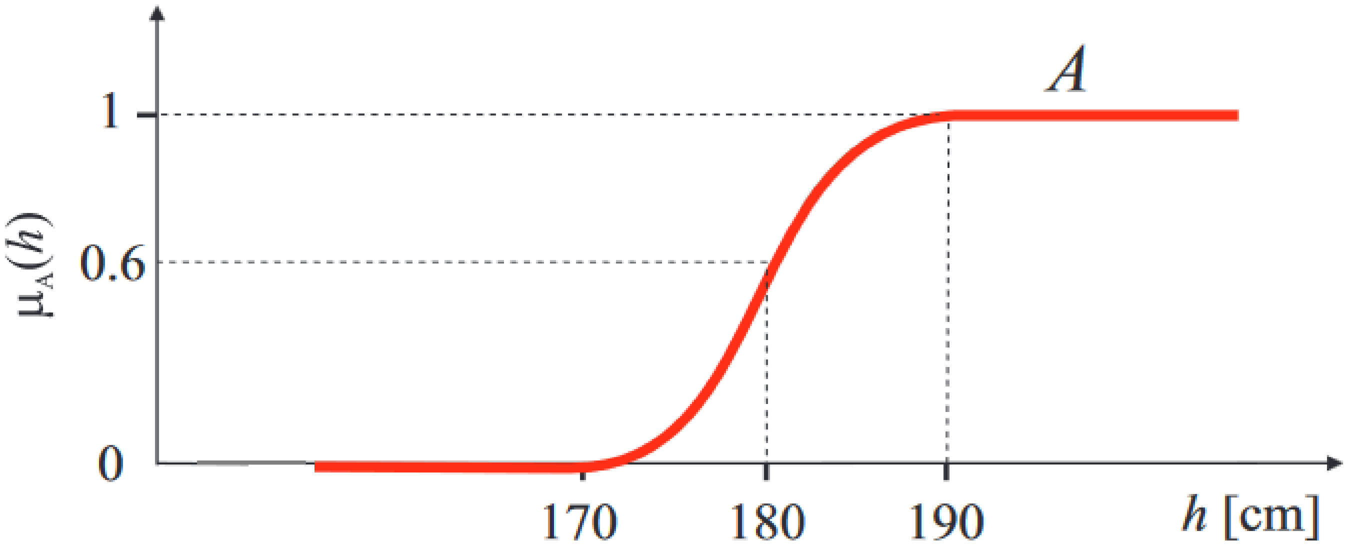
Fuzzy Logic Propositions

"John is tall" **. . . degree of truth**

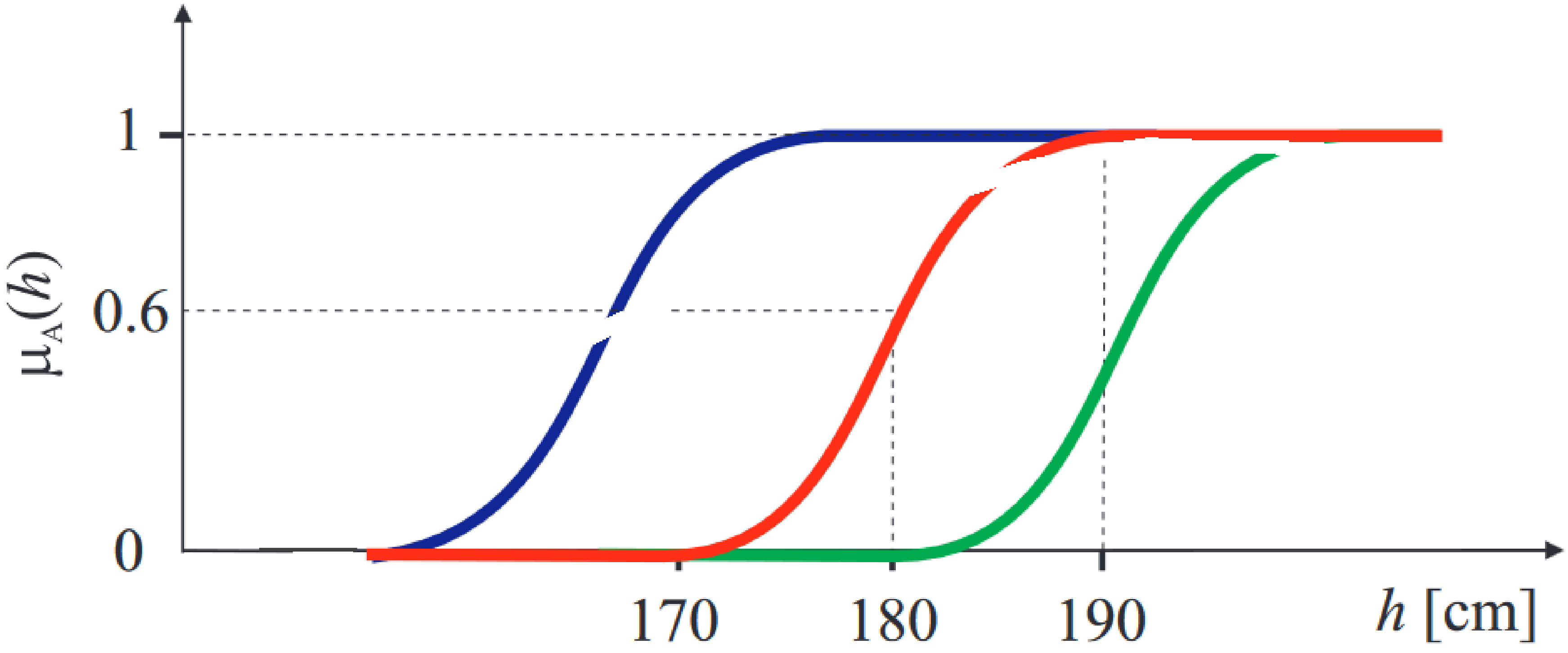
John's height: $h_{John} = 180.0$ $\mu_A(180.0) = 0.6$

$h_{John} = 179.5$ $\mu_A(179.5) = 0.56$

$h_{Paul} = 201.0$ $\mu_A(201.0) = 1$



Based on Subjective and Context Dependent



tall in China

tall in Europe

tall in NBA

Fuzzy Set

Lets take an example the set of YOUNG PEOPLE.

- *A one year old baby will clearly be a member of the set, and a 100 years old person will not be a member of this set, but what about people at the age of 20, 30, or 40 years?*
- *Zadeh proposed a **degree of membership/grade of membership**, such that the transition from membership to non-membership is gradual rather than abrupt.*
- *The grade of membership for all its members thus describes a fuzzy set. An item's grade of membership is normally a real number between 0 and 1, often denoted by the Greek letter μ.*
- *There is no formal basis for how to determine the grade of membership. It depends on the user and on the context.*

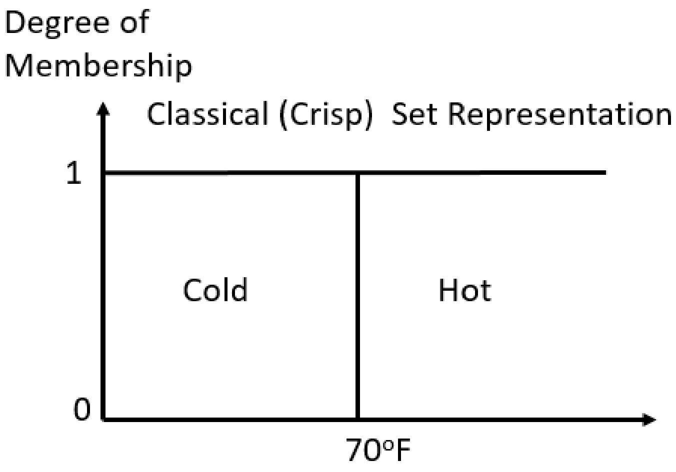
Fuzzy Set:

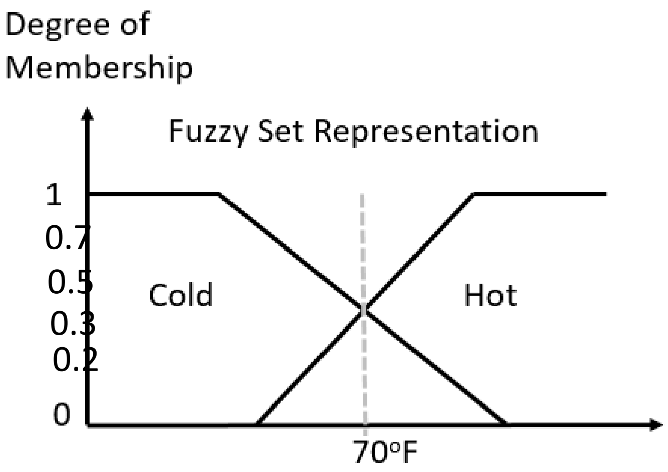
Classical Set:

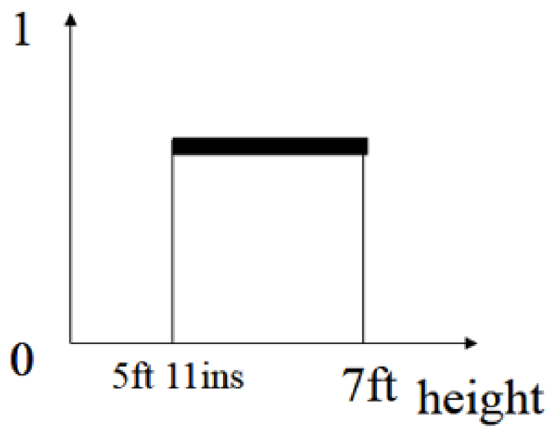
If the temperature is higher than 70°F, it is hot; otherwise, it is not hot.

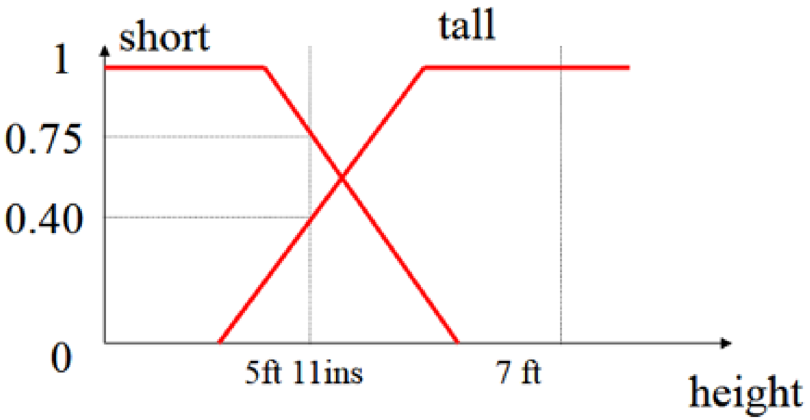
Cases:

– Temperature = 100°F Hot
– Temperature = 70.1°F Hot
– Temperature = 69.9°F Not hot
– Temperature = 50°F Not hot









Formal definitions of a fuzzy set:

- A fuzzy set is uniquely defined by its membership function
- A fuzzy set A in X is defined by a set of ordered pairs.

element

Universe of discourse

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

Fuzzy set

Membership function (MF)

- The MF maps every element of X to a membership value membership grade between 0 and 1.

Fuzzy set operations:

- The Union (Disjunction) of these two sets in terms of function-theoretic terms is given as follows (the symbol ∨ is the maximum):

  Union A U B : $\mu_{A \cup B}(x) = \mu_A(x) \vee \mu_B(x) = \max(\mu_A(x), \mu_B(x))$

- The Intersection(Conjunction) of these two sets in function-theoretic terms is given by (the symbol ∧ is the minimum operator):

  Intersection A ∩ B : $\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x) = \min(\mu_A(x), \mu_B(x))$

- The Complement (negation) of a single set on universe X, say A, is given by

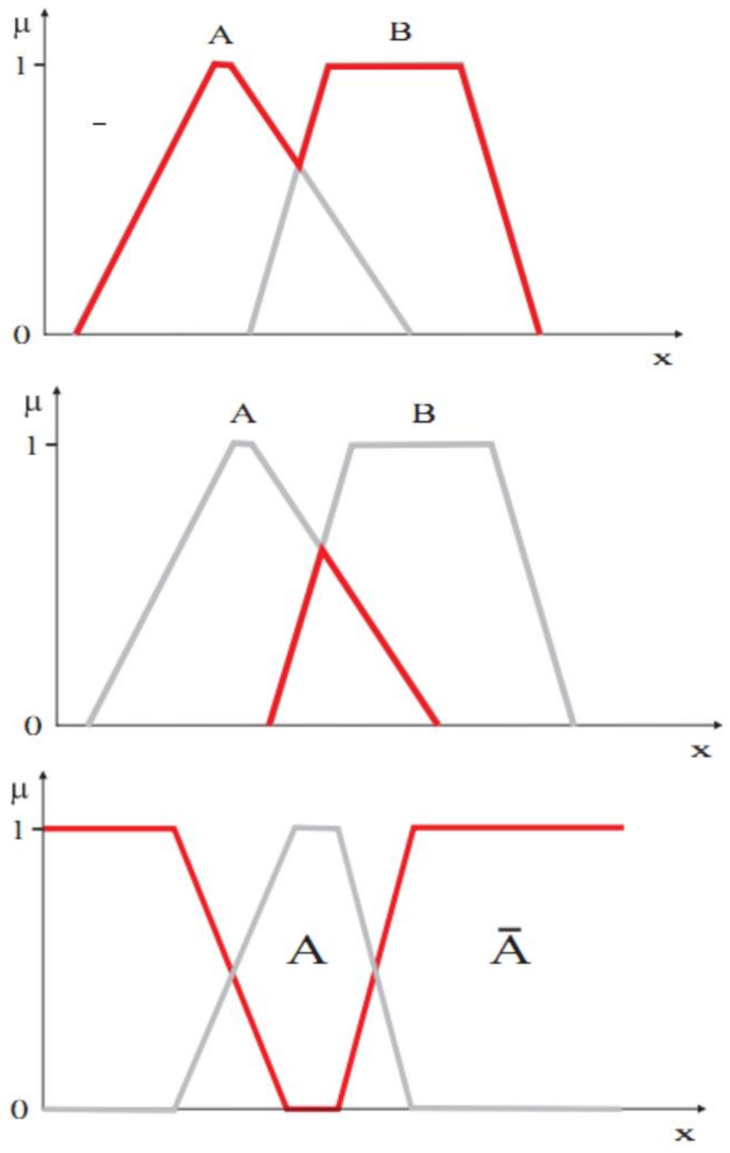
  complement(Ā): $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$

Fuzzy Set operations

$\mu_{A \cup B}(x) = \mu_A(x) \vee \mu_B(x) = \max(\mu_A(x), \mu_B(x))$

$\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x) = \min(\mu_A(x), \mu_B(x))$

$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$



Fuzzy set operations

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$ $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

Union:

$$A \cup B = \{(x_1,0.8),(x_2,0.7),(x_3,1)\}$$

Because

$$\mu_{A\cup B}(x_1) = \max(\mu_A(x_1), \mu_B(x_1))$$
$$= \max(0.5,0.8)$$
$$= 0.8$$

$$\mu_{A\cup B}(x_2) = 0.7 \text{ and } \mu_{A\cup B}(x_3) = 1$$

Fuzzy Set Operations

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$ $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

Intersection:

$$A \cap B = \{(x_1,0.5),(x_2,0.2),(x_3,0)\}$$

Because

$$\mu_{A\cap B}(x_1) = \min(\mu_A(x_1), \mu_B(x_1))$$

$$= \max(0.5,0.8)$$

$$= 0.5$$

$$\mu_{A\cap B}(x_2) = 0.2 \text{ and } \mu_{A\cap B}(x_3) = 0$$

Fuzzy Set Operations

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$

Complement:

$$A^c = \{(x_1,0.5),(x_2,0.3),(x_3,1)\}$$

Because

$$\mu_{\bar{A}}(x_1) = 1 - \mu_A(x_1)$$

$$= 1 - 0.5$$

$$= 0.5$$

$$\mu_A(x_2) = 0.3 \text{ and } \mu_A(x_3) = 1$$

Fuzzy set

Containment/Subset: The fuzzy set A is contained in the fuzzy set B (or A is a subset of B) if and only if $\mu_A \leq \mu_B$.

We will denote subset between two fuzzy sets as $A \subseteq B$.

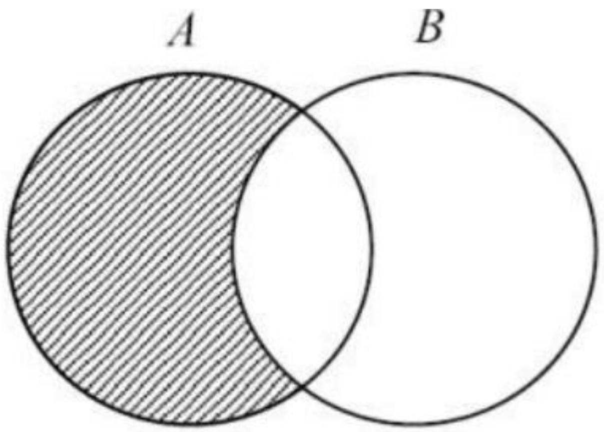
From this definition of containment, it can easily be seen that for two fuzzy sets A and B that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

The notion of a proper subset will be the same as the definition of subset but with $\mu_A < \mu_B$ and be denoted as $A \subset B$.

Empty Fuzzy Set : A fuzzy set A is empty if and only if its membership function is identically zero on X, $\mu_A \equiv 0$.

Fuzzy Set Equality : Two fuzzy sets A and B are equal if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$. We will denote equality as $A = B$.

Other Fuzzy set operations



- Difference
  - Crisp set $A - B = A \cap \overline{B}$
  - Fuzzy set : Simple difference

    By using standard complement and intersection operations. $A - B = A \cap \overline{B}$

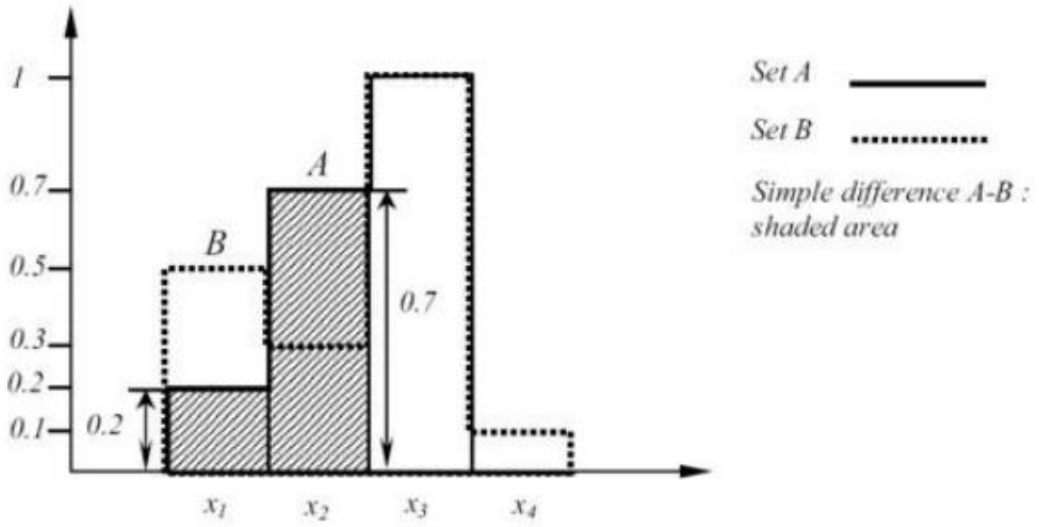
  - Fuzzy set : Bounded difference

    $\mu_{A\theta B}(x) = Max[0, \mu_A(x) - \mu_B(x)]$

Other Fuzzy set operations

- Example
  - Simple difference

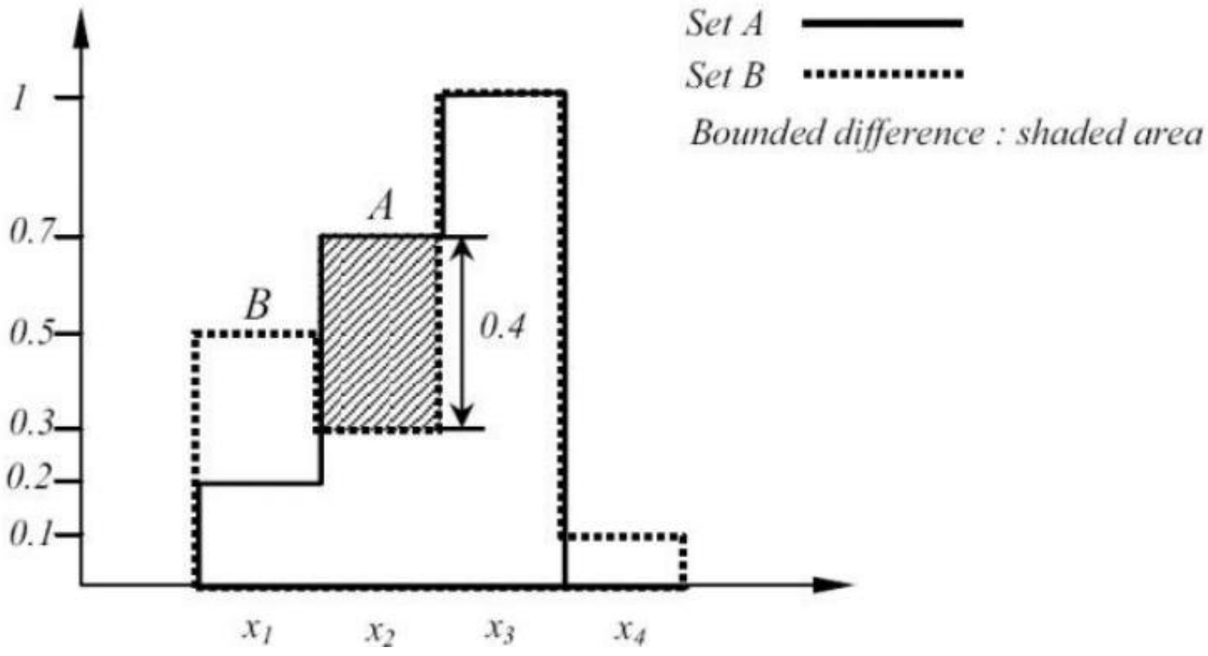
$$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$$
$$B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$$
$$\overline{B} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.9)\}$$
$$A - B = A \cap \overline{B} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0), (x_4, 0)\}$$



Other Fuzzy set operations

- Example
  - Bounded difference $\mu_{A\theta B}(x) = \text{Max}[0, \mu_A(x) - \mu_B(x)]$

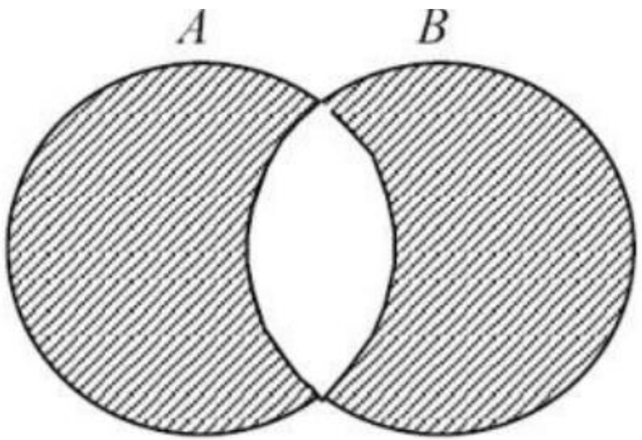
$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$
$B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$
$A\ \theta\ B = \{(x_1, 0), (x_2, 0.4), (x_3, 0), (x_4, 0)\}$



Other Fuzzy set operations

- **Disjunctive sum (exclusive OR)**

$$A \oplus B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$$



$$\mu_{\overline{A}}(x) = 1 - \mu_A(x), \quad \mu_{\overline{B}}(x) = 1 - \mu_B(x)$$

$$\mu_{A \cap \overline{B}}(x) = Min[\mu_A(x), 1 - \mu_B(x)]$$

$$\mu_{\overline{A} \cap B}(x) = Min[1 - \mu_A(x), \ \mu_B(x)]$$

$$\mathrm{A} \oplus \mathrm{B} = (A \cap \overline{B}) \cup (\overline{A} \cap B), \text{ then}$$

$$\mu_{A \oplus B}(x) = Max\{Min[\mu_A(x), 1 - \mu_B(x)], \ Min[1 - \mu_A(x), \mu_B(x)]\}$$

Other Fuzzy set operations

Example 2.2 Here goes procedures obtaining disjunctive sum of A and B (Fig 2.12).

$$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$$
$$B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$$
$$\overline{A} = \{(x_1, 0.8), (x_2, 0.3), (x_3, 0), (x_4, 1)\}$$
$$\overline{B} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.9)\}$$
$$A \cap \overline{B} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0), (x_4, 0)\}$$
$$\overline{A} \cap B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0), (x_4, 0.1)\}$$

and as a consequence,

$$A \oplus B = (A \cap \overline{B}) \cup (\overline{A} \cap B) = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.1)\} \quad \square$$

Other Fuzzy set operations

- Distance
  - Hamming distance

$$d(A, B) = \sum_{i=1,\, x_i \in X}^{n} \left| \mu_A(x_i) - \mu_B(x_i) \right|$$

$$A = \{(x_1, 0.4), (x_2, 0.8), (x_3, 1), (x_4, 0)\}$$
$$B = \{(x_1, 0.4), (x_2, 0.3), (x_3, 0), (x_4, 0)\}$$

Hamming distance; $d(A, B)$,

$$d(A, B) = |0| + |0.5| + |1| + |0| = 1.5$$

  - Euclidean distance

$$A = \{(x_1, 0.4), (x_2, 0.8), (x_3, 1), (x_4, 0)\}$$
$$B = \{(x_1, 0.4), (x_2, 0.3), (x_3, 0), (x_4, 0)\}$$

$$e(A, B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x) - \mu_B(x))^2}$$

$$e(A, B) = \sqrt{0^2 + 0.5^2 + 1^2 + 0^2} = \sqrt{1.25} = 1.12$$

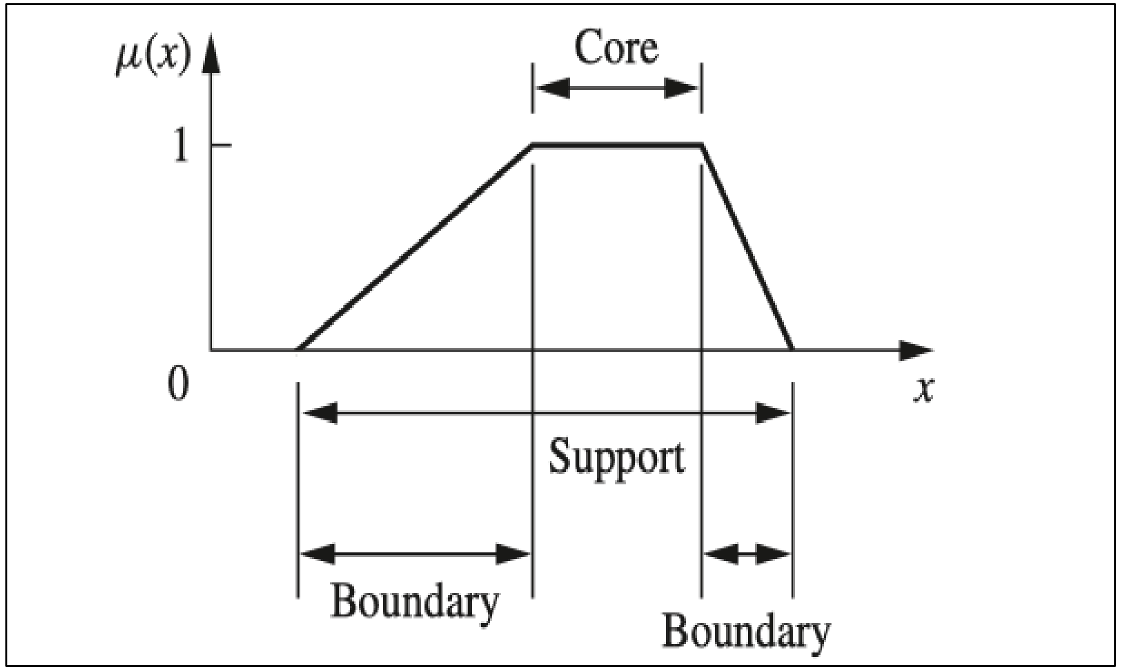
Support, core, boundary

The Support of fuzzy set:
The support of a fuzzy set (denoted supp) is the crisp set of all $x \in X$ for which $\mu_A(x) > 0$

The (core) of a fuzzy set:
is the crisp set for which $\mu_A(x) = 1$

The (boundary) of a fuzzy set:
is the crisp set for which $0 < \mu_A(x) < 1$

Normal fuzzy set:
A fuzzy subset A of the universal set X is called normal if there exists an $x \in X$ such that $\mu_A(x) = 1$. Otherwise A is subnormal.



Fuzzy sets:

Fuzzy singleton set:

Fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called fuzzy singleton

Fuzzy crossover point:

Crossover point of a fuzzy set A is a point x in X such that $\{(x|\ \mu_A(x) = 0.5\}$

α-cut :

α−cut of a fuzzy set A is set of all points x in X such that $\{(x|\ \mu_A(x) \geq \alpha\ \}$

Convexity:

Convexity $\mu_A(\lambda x_1 + (1\text{-}\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$. Then A is convex, where $\lambda \in [0,1]$

Bandwidth:

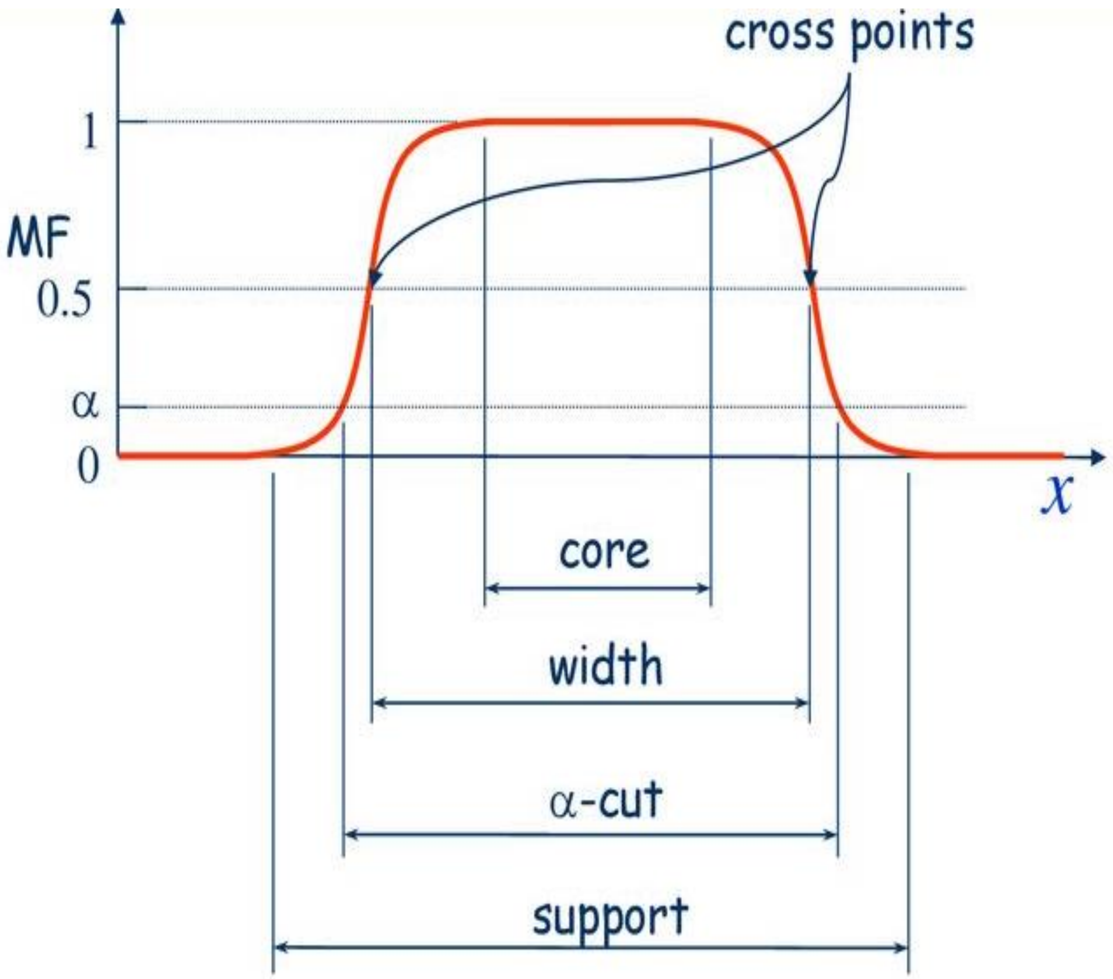
Bandwidth(A)= $|x_2\text{-}x_1|$, where $x_2 = 0.5$ and $x_1 = 0.5$ are crossover points.

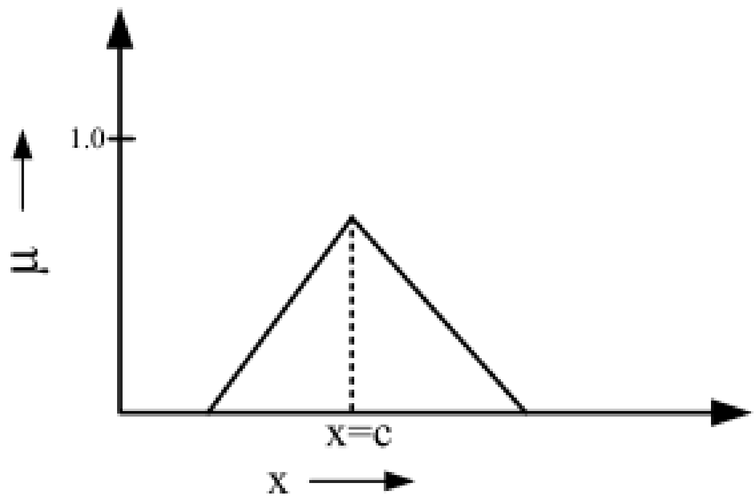
Symmetry:

Symmetry $\mu_A(c+x) = \mu_A(c-x)$ for all $x \in X$. Then A is symmetric.

Fuzzy numbers:

A fuzzy number A is a fuzzy set in the real line R that satisfies the conditions for normality and convexity.





Symmetry:

Q) Let X = {a, b, c, d} and fuzzy set A = ${}^{0.8}/_{a} + {}^{1.0}/_{b} + {}^{0.3}/_{c} + {}^{0.1}/_{d}$

(i) α -cut sets for α = 0.1, 0.3, 0.8, 1.0

(ii) Strong α -cut sets for α = 0.1, 0.3, 0.8, 1.0

Example (Normality) : Let A = ${}^{0.2}/_{1} + {}^{0.5}/_{2} + {}^{0.8}/_{3} + {}^{1}/_{4} + {}^{0.7}/_{5} + {}^{0.3}/_{6}$, Since $\mu_A(4) = 1$, then this fuzzy set is normal.

While the fuzzy set A = ${}^{0.2}/_{1} + {}^{0.5}/_{2} + {}^{0.8}/_{3} + {}^{0.7}/_{5} + {}^{0.3}/_{6}$ is subnormal.

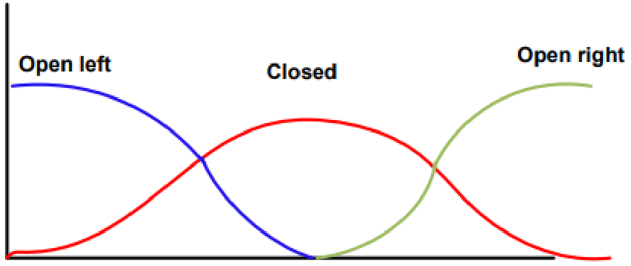
Fuzzy sets:

- Open left, open right, closed: A fuzzy set is open left if

Open left : If $\lim_{x \to -\infty} \mu_A(x) = 1$ and $\lim_{x \to +\infty} \mu_A(x) = 0$

Open right : If $\lim_{x \to -\infty} \mu_A(x) = 0$ and $\lim_{x \to +\infty} \mu_A(x) = 1$

Closed If : $\lim_{x \to -\infty} \mu_A(x) = \lim_{x \to +\infty} \mu_A(x) = 0$



- Height :The maximum value of the membership is called *height.*

  Example: Let $A = {}^{0.8}/_{1} + {}^{0.5}/_{2} + {}^{0.9}/_{3} + {}^{0.1}/_{4}$. the height of this fuzzy set is 0.9

- Containment or Subset: Fuzzy set A is contained in fuzzy set B (or A is a subset of B) if $\mu_A(x) \leq \mu_B(x)$ for all x.

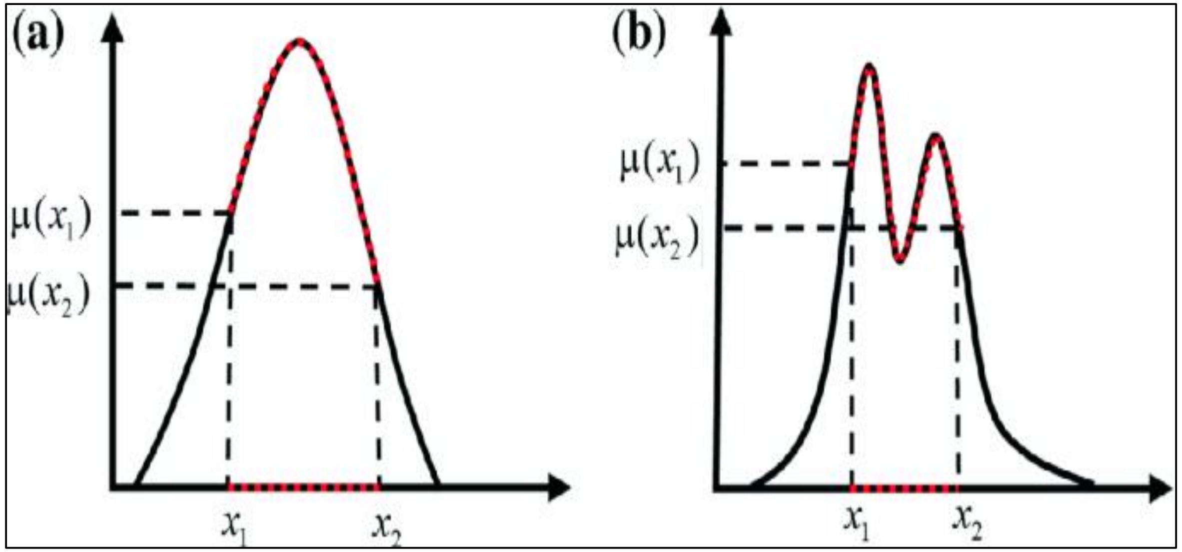
Convexity:

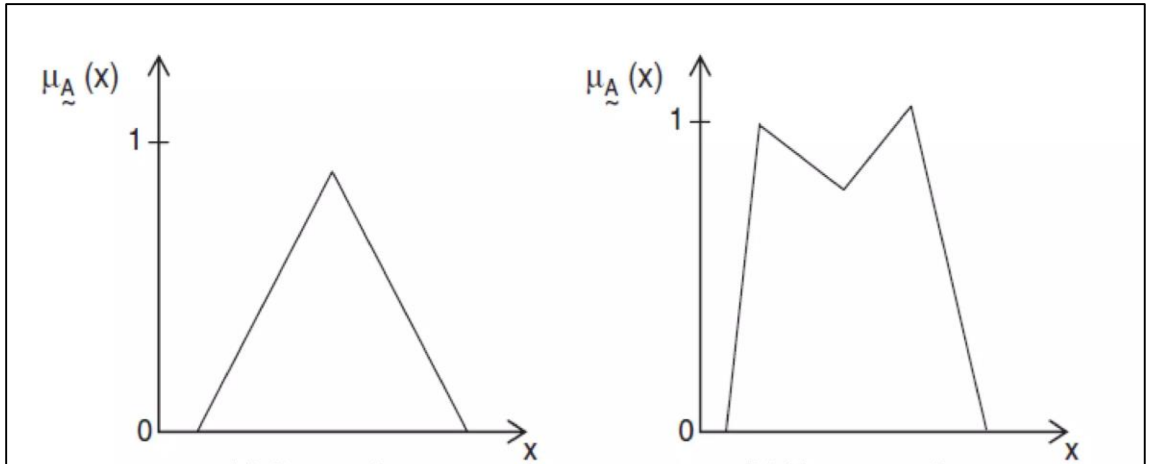
Fuzzy set A is convex if for any $\lambda \in [0,1]$.

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

i.e. for any element $t = \mu_A(\lambda x_1 + (1-\lambda)x_2)$ in between x_1 and x_2 in a fuzzy set A, the relation $x_1 < t < x_2$ implies that

$$t \geq \min(\mu_A(x_1), \mu_A(x_2))$$





(a) Convex fuzzy set

(b) Nonconvex fuzzy set

Fuzzy convexity

A convex fuzzy set is described by a membership function whose membership values are strictly monotonically increasing, or whose membership values are strictly monotonically decreasing, or whose membership values are strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe.

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

- How do you distinguish a fuzzy set from a crisp set?

Answer:

- Do the Laws of Contradiction and Excluded Middle hold?

Answer: However, if A is a non-crisp set, then neither law will hold. Indeed, note that for a non-crisp set, there exists some $x \in A$ such that $\mu_A(x) \in (0, 1)$, i.e. $\mu_A(x) = 0, 1)$ Thus, we have

- $\mu_{A \cap A'}(x) = \max\{\mu_A(x) , 1- \mu_A(x) \} ǂ 0$
- $\mu_{A \cup A'}(x) = \min\{\mu_A(x) , 1- \mu_A(x) \} ǂ 1$

Cardinality

- Cardinality of fuzzy set A, is $|A| = \sum_{i}^{n} \mu_A(x)$
- Relative cardinality of set A is : $||A|| = {}^{|A|}/_{|X|}$

Example : Let X = {1,2,3, 4,5,6, 7, 8, 9, 10}

A = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6

B = 0.3/1 + 0.9/3 + 0.8/6 + 0.1/7 + 0.4/8 + 0.6/9 + 1/10

Then find the fuzzy cardinality?

Cartesian product and co-product

- Let A and B be fuzzy sets in X and Y, respectively. The Cartesian product of A and B, denoted by A x B, is a fuzzy set in the product space X x Y with membership function

$$\mu_{A\times B}(x,y) = \min(\mu_A(x), \mu_B(y)).$$

- Cartesian co-product A+B is fuzzy set with the membership function

$$\mu_{A+B}(x,y) = \max(\mu_A(x), \mu_B(y)).$$

Difference between Fuzzy and Crisp set

Fuzzy Set

- Fuzzy set have gradual transition from membership to non-membership or vice versa with many membership degrees/values.
- Infinite number of values: [0,1]
- Fuzzy controller
- Gradual membership
- Ambiguous boundary or not well defined boundary

Classical/ Crisp set

- In classical/crisp set the transition is abrupt/sudden not gradual.
- Bi-valued : {0,1}
- Digital system design
- Total membership
- Precise or well defined boundary

Overview (Fuzzy Set)

- A fuzzy set is an extension of the concept of a classical set whereby objects can be assigned partial membership of a fuzzy set; partial membership is not allowed in classical set theory.
- The degree an object belongs to a fuzzy set, a real number between 0 and 1, is called the membership value of the set.
- The meaning of a fuzzy set, is thus characterized by a membership function that maps elements of a universe of discourse to their corresponding membership values. The membership function of a fuzzy set A is denoted as μ.
- A fuzzy set is often denoted by its membership function

- Many authors denote the membership grade $\mu_A(x)$ by $A(x)$.

$$\mu_A(x) = A(x).$$

Assignment-1

Consider a universal set X which is defined on the age domain.

X := {5, 15, 25, 35, 45, 55, 65, 75, 85}

Answer the following:

1. Find fuzzy sets such as *infant, young, adult* and *senior* in X
2. Find the *support* set of each fuzzy set.
3. Find the *α-cut set* is derived from fuzzy set *young*.
4. Is the fuzzy set *adult* is normal or subnormal?
5. What is the height of fuzzy set senior

age(element)	infant	young	adult	senior
5	0	0	0	0
15	0	0.2	0.1	0
25	0	1	0.9	0
35	0	0.8	1	0
45	0	0.4	1	0.1
55	0	0.1	1	0.2
65	0	0	1	0.6
75	0	0	1	1
85	0	0	1	1

Assignment-2

Consider of the fuzzy sets A, B and C defined on X = [0, 10] of real numbers by the membership grade functions

(i) $\mu_A(x) = \frac{1}{1+x^2}$

(ii) $\mu_B(x) = 2^x$

(iii) $\mu_C(x) = \frac{x}{x+2}$

Q1) Calculate: (i) A', B', C' ; (ii) A U B, A U C, BU C; (iii) A U B U C ;(iv) A ∩ B ∩ C; (v) A ∩ C' ; (vi) $\overline{A \cap B}$;

Q2) Calculate the α-cuts and strong α-cuts of the three fuzzy in the sets for some values of α, for example α = 0.2, 0.1, 0.4, 1.0.

Assignment-3

(i) How do you distinguish a fuzzy set from a crisp set?

(ii) What is the difference between Fuzzy set and Probability ?

Difference between Fuzzy set and Probability ?

- The key difference between fuzzy logic and probability is how they handle uncertainty
  - Fuzzy Sets: Deal with vagueness and imprecision.
  - Probability: Deals with randomness and unpredictability
- Fuzziness describes event ambiguity. It measures the degree to which an event occurs, NOT whether it occurs.
- Randomness describes the uncertainty of event occurrence, an event occurs or not.
- Fuzzy Sets: Use a membership function to assign degrees of membership.
- Probability: Use a probability function to assign likelihoods to events.
- Fuzzy Sets: The degrees of membership of elements in a set do not need to sum to 1.
- Probability: The probabilities of all possible outcomes in a probability space must sum to 1
- Fuzzy Sets: Used in scenarios requiring human-like reasoning and dealing with imprecise information.
- Probability: Used in scenarios involving statistical analysis and prediction of future events based on known data.

(Fuzziness describes the lack of distinction of an event, whereas chance describes the uncertainty in the occurrence of the event. The event will occur or not occur; but is the description of the event clear enough to measure its occurrence or nonoccurrence?)

**** Fuzziness is a type of deterministic uncertainty*

Questions

We want to compare two sensors based upon their detection levels and gain settings. For a universe of discourse of gain settings, $X = \{0, 20, 40, 60, 80, 100\}$, the sensor detection levels for the monitoring of a standard item provides typical membership functions to represent the detection levels for each of the sensors; these are given below in standard discrete form:

$$\underset{\sim}{S}_1 = \left\{ \frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} + \frac{1.0}{80} + \frac{1.0}{100} \right\}$$

$$\underset{\sim}{S}_2 = \left\{ \frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} + \frac{0.95}{80} + \frac{1.0}{100} \right\}$$

Find the following membership functions using standard fuzzy operations:

(*a*) $\mu_{\underset{\sim}{S}_1 \cup \underset{\sim}{S}_2}(x)$

(*b*) $\mu_{\underset{\sim}{S}_1 \cap \underset{\sim}{S}_2}(x)$

(*c*) $\mu_{\overline{\underset{\sim}{S}_1}}(x)$

(*d*) $\mu_{\overline{\underset{\sim}{S}_2}}(x)$

(*e*) $\mu_{\overline{\underset{\sim}{S}_1} \cup \underset{\sim}{S}_1}(x)$

(*f*) $\mu_{\overline{\underset{\sim}{S}_1} \cap \underset{\sim}{S}_1}(x)$

Questions

$$\underset{\sim}{A} = \left\{ \frac{0.1}{0} + \frac{0.4}{1} + \frac{1}{2} + \frac{0.3}{3} + \frac{0.2}{4} \right\}$$

$$\underset{\sim}{B} = \left\{ \frac{0.2}{0} + \frac{0.5}{1} + \frac{1}{2} + \frac{0.4}{3} + \frac{0.1}{4} \right\}$$

Find the following:

(*a*) $\underset{\sim}{A} \cup \underset{\sim}{B}$

(*b*) $\underset{\sim}{A} \cap \underset{\sim}{B}$

(*c*) $\overline{\underset{\sim}{A}}$

(*d*) $\overline{\underset{\sim}{B}}$

(*e*) $\overline{\underset{\sim}{A} \cap \underset{\sim}{B}}$

(*f*) $\overline{\underset{\sim}{A} \cup \underset{\sim}{B}}$

Questions

$$\underset{\sim}{D}_1 = \left\{ \frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$$

$$\underset{\sim}{D}_2 = \left\{ \frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0} \right\}$$

For these two fuzzy sets, find the following:

(*a*) $\underset{\sim}{D}_1 \cup \underset{\sim}{D}_2$

(*b*) $\underset{\sim}{D}_1 \cap \underset{\sim}{D}_2$

(*c*) $\overline{\underset{\sim}{D}_1}$

(*d*) $\overline{\underset{\sim}{D}_2}$

(*e*) $\underset{\sim}{D}_1 \mid \underset{\sim}{D}_2$

(*f*) $\overline{\underset{\sim}{D}_1 \cup \underset{\sim}{D}_2}$

Questions

$$\underset{\sim}{A} = \left\{ \frac{0.15}{50} + \frac{0.25}{100} + \frac{0.5}{150} + \frac{0.7}{200} \right\} \quad \underset{\sim}{B} = \left\{ \frac{0.2}{50} + \frac{0.3}{100} + \frac{0.6}{150} + \frac{0.65}{200} \right\}$$

Calculate the union, intersection, and the difference